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Fractional integral Ostrowski inequality on L_P,0<P<∞ spaces

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Abstract: Recently we proved a type of Ostrowski inequality in terms of the quasi norm of the first derivative. Here we generalize this inequality type for fractional integral.
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Classification: *60E15; 39B72.*

1. Introduction

Firstly, let us recall and introduce some definitions notations, that we need in our work.

The function $f:[a,b] \subset R \to R$ is said to be convex if the following inequality holds
$f(tx+(1-t)y) \leq tf(x)+(1-t)f(y$ [4]
For all $x,y \in [a,b]$ and $t \in [0,1]$, we say that f is concave if$(-f)$ is convex.
$R^{\propto}$is the set of real numbers, $R^{\propto} = Q^{\propto} \cup J^{\propto}$,where $Q^{\propto}$ is the $\alpha-$type set of the rational numbers is defined as the set
$\left\{m^{\propto} = \left(\frac{p}{q}\right)^{\alpha}, p, q\epsilon Z, q \neq 0\right\}$,
also $J^{\propto}$ is the $\alpha-$type set of the irrational numbers is defined as the set

$$\left\{m^{\propto} \neq \left(\frac{P}{q}\right)^{\alpha}, p, q\epsilon Z, q \neq 0\right\}.$$

The local fractional derivative of $f(x)$ of order α at $X = X_\circ$ is defined as:

$$f^{\alpha}(X_\circ) = \frac{d^{\alpha}f(x)}{dx^{\alpha}}\bigg|_{X=X_\circ} = \lim_{X \to X_\circ} \frac{\Delta^{\alpha}(f(X)-f(X_\circ))}{(X-X_\circ)}$$

Where $\Delta^{\alpha}(f(X)-f(X_\circ)) \cong \Gamma(\alpha+1)\big(f(X)-f(X_\circ)\big)$.
If there exists $f^{(k+1)\alpha}(x) = D_x^{\alpha} \dots D_x^{\alpha} f(x)$ mm times for any $X \in I \subseteq R$,then we denoted $f \in D_{(k+1)\alpha}(I)$, where $k = 0,1,\dots$ [5]

Anon –differentiable function $f: R \to R^{\alpha}, X \to f(x)$ is called to be local fractional continuous at $X_\circ$.If for any $\in > 0$, there exists $\delta > 0$, such that $|f(x)-f(X_\circ)| < \in^{\propto}$ holds for $|X-X_\circ| < \delta$, where $\in, \delta \in R$. If $f(x)$ is local continuous on the interval (a,b).We denote $f(x) \in C_{\alpha}(a,b)$. [5]
We call f is fractional integrable if

$$\frac{1}{\Gamma(1+\alpha)}\int_a^b f(x)(dt)^{\alpha} = \frac{1}{\Gamma(1+\alpha)} \lim_{\Delta t \to 0} \sum_{j=0}^{n-1} f(t_j)\left(\Delta t_j\right)^{\alpha} < \infty \quad (1)$$

and the fractional integrable defined by:

$$aI_b^{\propto} f(x) = \frac{1}{\Gamma(1+\alpha)}\int_a^b f(x)(dt)^{\alpha} = \frac{1}{\Gamma(1+\alpha)} \lim_{\Delta t \to 0} \sum_{j=0}^{n-1} f(t_j)\left(\Delta t_j\right)^{\alpha}$$

With $\Delta t_j = t_{j+1} - t$,and $\Delta t = max\{\Delta t_1, \Delta t_2, \dots, \Delta t_{n-1}\}$,where $[t_j, t_{j+1}]$,

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$j = 0, \dots, n-1$, and $a = t_0 < t_1 < \cdots < t_{n-1} = b$ is a partition of interval $[a,b]$.Here it follows that $aI_b^{\propto} f(x) = 0$ if $a = b$ and $aI_b^{\propto} f(x) = -b\ aI_b^{\propto} f(x)$ if $a < b$.
Let $f: I \subset R \to R^{\alpha}$.For any $X_1, X_2 \in I$and $\lambda \in [0,1]$, if the following inequality $f(\lambda X_1) + (1-\lambda)X_2 \leq \lambda^{\alpha} f(X_1) + (1-\lambda)^{\alpha} f(X_2)$ holds, then f is called a generalized convex function on I.
Now, let us introduce our $L_{P,\propto}$ space for $0 < P < \infty$.

Let us define the fractional integrable quasi normed space as:

$$L_{P,\propto}[a,b] = \left\{ f: [a,b] \to R: \|f\|_{P,\alpha} = \left(\int_a^b |f(x)|^p \,(dx)^{\alpha} \right)^{\frac{1}{P}} < \infty \right\}$$

and $\|.\|_{P,\alpha}$ is a fractional L_P integrable norm.

2. Auxilary Results

Lemma 2.1 [6]:

$$\frac{d^{\alpha} f(x)}{dx^{\alpha}} = \frac{\Gamma(1+k\alpha)}{\Gamma(1+(k-1)\alpha)} X^{(k-1)\alpha}$$

$$\frac{1}{\Gamma(1+\alpha)} \int_a^b X^{k\alpha} (dx)^{\alpha} = \frac{\Gamma(1+k\alpha)}{\Gamma(1+(k+1)\alpha)} \left(b^{(k+1)\alpha} - a^{(k+1)\alpha} \right), k \epsilon R.$$

Lemma 2.2 [1]: Generalized Holder's inequality
Let $f, g \in C_{\alpha}[a,b], p, q > 1$, with $\frac{1}{p} + \frac{1}{q} = 1$, then

$$\frac{1}{\Gamma(1+\alpha)} \int_a^b |f(x) g(x)| (dx)^{\alpha} \leq \left(\frac{1}{\Gamma(1+\alpha)} \int_a^b |f(x)|^p (dx)^{\alpha} \right)^{\frac{1}{p}} . \left(\frac{1}{\Gamma(1+\alpha)} \int_a^b |g(x)|^q (dx)^{\alpha} \right)^{\frac{1}{q}}.$$

Lemma 2.3 [1]: in L_P-space if If $p < q$, then

$$\left(\sum_{i=1}^{\infty} |x_i|^q \right)^{\frac{1}{q}} \leq \left(\sum_{i=1}^{\infty} |x_i|^p \right)^{\frac{1}{p}}.$$

Lemma 2.4 [2]: Generalized Montgomery inequality

Let $I \subset R$ be an interval,$f: I^{\circ} \subset R \to R^{\alpha}$ ($I^{\circ}$is the interior of I) such that f is $\propto -$integrable for $a, b \in I^{\circ}$ with $a < b$.Then we have the identity

$$f(x) - \frac{\Gamma(1+\alpha)}{(b-a)^{\alpha}} aI_b^{\propto} f(x) = \frac{1}{\Gamma(1+\alpha)} \int_a^b p(x,t) f(x)^{\alpha} (dt)^{\alpha}$$

Where

$$p(x,t) = \begin{cases} (t-a)^{\alpha}, t \in [a,x] \\ (t-b)^{\alpha}, t \in [x,b] \end{cases}$$

Lemma 2.5 [6]: A second type generalized Montgomery inequality

Let $I \subset R$ be an interval,$f: I^{\circ} \subset R \to R^{\alpha}$($I^{\circ}$is the interior of I) such that $f \in D_{\alpha}(I^{\circ})$ and f is $\propto -$integrable with $a < b$,then

$$(I-h)^{\alpha} f(x) + h^{\alpha} \frac{f(a)+f(b)}{2^{\alpha}} - \frac{\Gamma(1+\alpha)}{(b-a)^{\alpha}} aI_b^{\propto} f(x) = \frac{1}{\Gamma(1+\alpha)} \int_a^b p(x,t) f(x)^{\alpha} (dt)^{\alpha}.$$

Where

$$p(x,t)=\begin{cases} t-\left(a+h\left(\frac{b-a}{2}\right)\right)^{\propto}, t\in[a,x] \\ t-\left(b-h\left(\frac{b-a}{2}\right)\right)^{\propto}, t\in[x,b] \end{cases}.$$

Where $h\in[0,1]$ and $a+h\left(\frac{b-a}{2}\right)\le x\le b-h\left(\frac{b-a}{2}\right)$.

3. Main Results

Let us now introduce our main results. We use two kinds of generalized Montgomery identity to prove types generalized Ostrowski Theorems.

Theorem 3.1:

If $f\subset R, f: I^{\circ}\subset R\to R^{\alpha}$ be a map $[a,b]\subset I^{\circ}$ and $f\in L_{P,\alpha}[a,b]$.

Then

$$(1)\left|f(x)-\frac{\Gamma(1+\alpha)}{(b-a)^{\alpha}}aI_b^{\propto}f(x)\right|\le\frac{\left(\Gamma(1+\alpha q)\right)^{\frac{1}{q}}(b-a)^{\frac{\propto}{q}}}{\left(\Gamma(1+(q+1)\alpha)\right)^{\frac{1}{q}}\left(\Gamma(1+\alpha)\right)^{\frac{1}{p}}}\|f^{\alpha}\|_p,$$

$$1\le p,q\le\infty,\frac{1}{p}+\frac{1}{q}=1.$$

$$(2)\left|f(x)-\frac{\Gamma(1+\alpha)}{(b-a)^{\alpha}}aI_b^{\propto}f(x)\right|$$

$$\le\frac{C(p)\left(\Gamma(1+\alpha q)\right)^{\frac{1}{q}}(b-a)^{\frac{\propto}{q}}}{\left(\Gamma(1+(q+1)\alpha)\right)^{\frac{1}{q}}\left(\Gamma(1+\alpha)\right)^{\frac{1}{p}}}\|f^{\alpha}\|_p,\qquad 0<p<1.$$

Where, I° is the interior of the interval I.

Proof: According to p, let us divide our proof into two cases.

Case1: $1\le p\le\infty$

by using the generalized Holder's inequality described in Lemma(2.2) , we get

$$\frac{1}{\Gamma(1+\alpha)(b-a)^{\alpha}}\int_a^b|p(x,t)||f(x)^{\alpha}|\quad(dt)^{\alpha}$$

$$\le\frac{1}{(b-a)^{\alpha}}\left(\frac{1}{\Gamma(1+\alpha)}\int_a^b|p(x,t)|^q(dx)^{\alpha}\right)^{\frac{1}{q}}\left(\frac{1}{\Gamma(1+\alpha)}\int_a^b|f(x)^{\alpha}|^p(dx)^{\alpha}\right)^{\frac{1}{p}}$$

$$1\le p,q\le\infty,\frac{1}{p}+\frac{1}{q}=1 \qquad (2)$$

Let us calculate $\left(\frac{1}{\Gamma(1+\alpha)}\int_a^b|p(x,t)|^q(dx)^{\alpha}\right)^{\frac{1}{q}}$, we have

$$\left(\frac{1}{\Gamma(1+\alpha)}\int_a^x|(t-a)|^q(dt)^{\alpha}\right)^{\frac{1}{q}}+\left(\frac{1}{\Gamma(1+\alpha)}\int_x^b|(t-b)|^q(dt)^{\alpha}\right)^{\frac{1}{q}}$$

$$=(I_1+I_2)^{\frac{1}{q}} \qquad (3)$$

By using Lemma 2,1,

$$I_1=\frac{1}{\Gamma(1+\alpha)}\int_a^x|(t-a)|^q(dt)^{\alpha}$$

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$$= \frac{\Gamma(1+\alpha q)}{\Gamma(1+(q+1)\alpha)}\left(x^{(q+1)\alpha} - a^{(q+1)\alpha}\right) \quad (4)$$

$$I_2 = \frac{1}{\Gamma(1+\alpha)}\int_x^b |(t-b)|^q (dt)^\alpha = \frac{\Gamma(1+\alpha q)}{\Gamma(1+(q+1)\alpha)}\left(b^{(q+1)\alpha} - x^{(q+1)\alpha}\right) \quad (5)$$

Put (4) and(5) in (3), we get

$$\left(\frac{1}{\Gamma(1+\alpha)}\int_a^b |p(x,t)|^q (dx)^\alpha\right)^{\frac{1}{q}}$$

$$= \left(\frac{\Gamma(1+\alpha q)}{\Gamma(1+(q+1)\alpha)}\left(b^{(q+1)\alpha} - a^{(q+1)\alpha}\right)\right)^{\frac{1}{q}}$$

$$\leq \left(\frac{\Gamma(1+\alpha q)}{\Gamma(1+(q+1)\alpha)}\right)^{\frac{1}{q}} (b-a)^{\frac{(q+1)\alpha}{q}} \quad (6)$$

Since $\left(\frac{1}{\Gamma(1+\alpha)}\int_a^b |f(x)^\alpha|^p (dx)^\alpha\right)^{\frac{1}{p}} = \left(\frac{1}{\Gamma(1+\alpha)}\right)^{\frac{1}{p}} \|f^\alpha\|_p$ (7)

Put (6) and (7) in (2), we get

$$\frac{1}{\Gamma(1+\alpha)(b-a)^\alpha}\int_a^b |p(x,t)||f(x)^\alpha| \quad (dt)^\alpha \leq \frac{\left(\Gamma(1+\alpha q)\right)^{\frac{1}{q}}(b-a)^{\frac{(q+1)\alpha}{q}}}{(b-a)^\alpha\left(\Gamma(1+(q+1)\alpha)\right)^{\frac{1}{q}}\left(\Gamma(1+\alpha)\right)^{\frac{1}{p}}} \|f^\alpha\|_p,$$

$$1 \leq p,q \leq \infty, \frac{1}{p}+\frac{1}{q} = 1$$

Now by using Lemma 2.4,we get

$$\left|f(x) - \frac{\Gamma(1+\alpha)}{(b-a)^\alpha} a I_b^\alpha f(x)\right| \leq \frac{\left(\Gamma(1+\alpha q)\right)^{\frac{1}{q}}(b-a)^{\frac{\alpha}{q}}}{\left(\Gamma(1+(q+1)\alpha)\right)^{\frac{1}{q}}\left(\Gamma(1+\alpha)\right)^{\frac{1}{p}}} \|f^\alpha\|_p.$$

The first case is proved.

Case2: $0 < P < 1$

By using the generalized Holder's inequality described in Lemma2.2 , we obtain.

$$\frac{1}{\Gamma(1+\alpha)(b-a)^\alpha}\int_a^b |p(x,t)||f(x)^\alpha| \quad (dt)^\alpha$$

$$\leq \frac{1}{(b-a)^\alpha}\left(\frac{1}{\Gamma(1+\alpha)}\int_a^b |p(x,t)|^q (dx)^\alpha\right)^{\frac{1}{q}}\left(\frac{1}{\Gamma(1+\alpha)}\int_a^b |f(x)^\alpha|^h (dx)^\alpha\right)^{\frac{1}{h}}$$

$$1 \leq p,q \leq \infty, \frac{1}{h}+\frac{1}{q} = 1$$

by using definition of the fractional integral in(1) and (6)

$$\frac{1}{\Gamma(1+\alpha)(b-a)^\alpha}\int_a^b |p(x,t)||f(x)^\alpha| \quad (dt)^\alpha$$

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$$\leq \left(\frac{\Gamma(1+\alpha q)}{\Gamma(1+(q+1)\alpha)}\right)^{\frac{1}{q}} (b-a)^{\frac{(q+1)\alpha}{q}} \times \left(\frac{1}{\Gamma(1+\alpha)}\sum_{i=1}^{n}|f^{\alpha}(t_i)|^{h}(\Delta t_i)^{\alpha}\right)^{\frac{1}{h}}$$

Where $0 < p < 1$ and by using (1), we get

$$\frac{1}{\Gamma(1+\alpha)(b-a)^{\alpha}}\int_a^b |p(x,t)||f(x)^{\alpha}| \quad (dt)^{\alpha} \leq$$

$$\frac{C(p)\big(\Gamma(1+\alpha q)\big)^{\frac{1}{q}}(b-a)^{\frac{\alpha}{q}}}{\big(\Gamma(1+(q+1)\alpha)\big)^{\frac{1}{q}}\big(\Gamma(1+\alpha)\big)^{\frac{1}{p}}}\|f^{\alpha}\|_p, \quad 0 < p < 1.$$

Now by using Lemma 2.4,we get

$$\left|f(x) - \frac{\Gamma(1+\alpha)}{(b-a)^{\alpha}} aI_b^{\propto} f(x)\right|$$

$$\leq \frac{C(p)\big(\Gamma(1+\alpha q)\big)^{\frac{1}{q}}(b-a)^{\frac{\alpha}{q}}}{\big(\Gamma(1+(q+1)\alpha)\big)^{\frac{1}{q}}\big(\Gamma(1+\alpha)\big)^{\frac{1}{p}}}\|f^{\alpha}\|_p, \quad 0 < p < 1.$$

The second case is proved.

Theorem 3.2:

If $f \subset R, f: I^{\circ} \subset R \to R^{\alpha}$ be a map $[a,b] \subset I^{\circ}, f \in L_{P,\alpha}[a,b]$.

Then

$$(1)\left|(I-h)^{\alpha} f(x) + h^{\alpha}\frac{f(a)+f(b)}{2^{\alpha}} - \frac{\Gamma(1+\alpha)}{(b-a)^{\alpha}} aI_b^{\propto} f(x)\right|$$

$$\leq \frac{\big(\Gamma(1+\alpha q)\big)^{\frac{1}{q}}(b-a)^{\frac{\propto}{q}}}{\big(\Gamma(1+(q+1)\alpha)\big)^{\frac{1}{q}}\big(\Gamma(1+\alpha)\big)^{\frac{1}{p}}}\|f^{\alpha}\|_p,$$

$$1 \leq p, q \leq \infty, \frac{1}{p} + \frac{1}{q} = 1.$$

$$(2)\left|(I-h)^{\alpha} f(x) + h^{\alpha}\frac{f(a)+f(b)}{2^{\alpha}} - \frac{\Gamma(1+\alpha)}{(b-a)^{\alpha}} aI_b^{\propto} f(x)\right|$$

$$\leq \frac{C(p)\big(\Gamma(1+\alpha q)\big)^{\frac{1}{q}}(b-a)^{\frac{\propto}{q}}}{\big(\Gamma(1+(q+1)\alpha)\big)^{\frac{1}{q}}\big(\Gamma(1+\alpha)\big)^{\frac{1}{p}}}\|f^{\alpha}\|_p \quad ,0 < p < 1.$$

Proof:

We take two cases to prove our Theorem.

Case1: $1 \leq p \leq \infty$

By using Lemma 2.5 , we obtain.

$$\left|(I-h)^{\alpha} f(x) + h^{\alpha}\frac{f(a)+f(b)}{2^{\alpha}} - \frac{\Gamma(1+\alpha)}{(b-a)^{\alpha}} aI_b^{\propto} f(x)\right|$$

$$= \frac{1}{\Gamma(1+\alpha)(b-a)^{\alpha}}\int_a^b |p(x,t)||f(x)^{\alpha}| \quad (dt)^{\alpha}$$

By using the generalized Holder's inequality described in Lemma2.1 , we obtain.

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$$\left|(I-h)^{\alpha} f(x)+h^{\alpha} \frac{f(a)+f(b)}{2^{\alpha}}-\frac{\Gamma(1+\alpha)}{(b-a)^{\alpha}} a I_b^{\propto} f(x)\right|$$

$$\leq \frac{1}{(b-a)^{\alpha}}\left(\frac{1}{\Gamma(1+\alpha)} \int_a^b |p(x,t)|^q (dx)^{\alpha}\right)^{\frac{1}{q}}\left(\frac{1}{\Gamma(1+\alpha)} \int_a^b |f(x)^{\alpha}|^h (dx)^{\alpha}\right)^{\frac{1}{h}}$$

$$1 \leq p, q \leq \infty, \frac{1}{h}+\frac{1}{q}=1 \tag{8}$$

Let us calculate $\left(\frac{1}{\Gamma(1+\alpha)} \int_a^b |p(x,t)|^q (dx)^{\alpha}\right)^{\frac{1}{q}}$, we have

$$\left(\frac{1}{\Gamma(1+\alpha)} \int_a^x \left|t-\left(a+h\left(\frac{b-a}{2}\right)\right)\right|^{\alpha q} (dt)^{\alpha}\right)^{\frac{1}{q}}+\left(\frac{1}{\Gamma(1+\alpha)} \int_x^b \left|t-\left(b-h\left(\frac{b-a}{2}\right)\right)\right|^{\alpha q} (dt)^{\alpha}\right)^{\frac{1}{q}}$$

$$=(M_1+M_2)^{\frac{1}{q}} \tag{9}$$

$$M_1=\frac{1}{\Gamma(1+\alpha)} \int_a^x \left|t-\left(a+h\left(\frac{b-a}{2}\right)\right)\right|^{\alpha q} (dt)^{\alpha}$$

$$=\frac{1}{\Gamma(1+\alpha)} \int_a^{a+h\left(\frac{b-a}{2}\right)} \left|t-\left(a+h\left(\frac{b-a}{2}\right)\right)\right|^{\alpha q} (dt)^{\alpha}+\frac{1}{\Gamma(1+\alpha)} \int_{a+h\left(\frac{b-a}{2}\right)}^{x} \left|t-\left(a+h\left(\frac{b-a}{2}\right)\right)\right|^{\alpha q} (dt)^{\alpha}$$

By using Lemma 2,1

$$M_1=\frac{\Gamma(1+\alpha q)}{\Gamma(1+(q+1)\alpha)}\left[\left(\left(a+h\left(\frac{b-a}{2}\right)\right)^{(q+1)\alpha}-a^{(q+1)\alpha}\right)+\left(x^{(q+1)\alpha}-\left(a+h\left(\frac{b-a}{2}\right)\right)^{(q+1)\alpha}\right)\right]$$

$$=\frac{\Gamma(1+\alpha q)}{\Gamma(1+(q+1)\alpha)}\left(x^{(q+1)\alpha}-a^{(q+1)\alpha}\right) \tag{10}$$

also,

$$M_2=\frac{1}{\Gamma(1+\alpha)} \int_x^b \left|t-\left(b-h\left(\frac{b-a}{2}\right)\right)\right|^{\alpha q} (dt)^{\alpha}$$

$$=\frac{1}{\Gamma(1+\alpha)} \int_a^{b-h\left(\frac{b-a}{2}\right)} \left|t-\left(b-h\left(\frac{b-a}{2}\right)\right)\right|^{\alpha q} (dt)^{\alpha}+\frac{1}{\Gamma(1+\alpha)} \int_{a+h\left(\frac{b-a}{2}\right)}^{x} \left|t-\left(b-h\left(\frac{b-a}{2}\right)\right)\right|^{\alpha q} (dt)^{\alpha}$$

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$$M_2=\frac{\Gamma(1+\alpha q)}{\Gamma(1+(q+1)\alpha)}\left[\left(\left(b-h\left(\frac{b-a}{2}\right)\right)^{(q+1)\alpha}-x^{(q+1)\alpha}\right)+\left(x^{(q+1)\alpha}-\left(b-h\left(\frac{b-a}{2}\right)\right)^{(q+1)\alpha}\right)\right]$$

$$=\frac{\Gamma(1+\alpha q)}{\Gamma(1+(q+1)\alpha)}\left(b^{(q+1)\alpha}-x^{(q+1)\alpha}\right) \quad (11)$$

Put (10) and(11) in (9), we get

$$\left(\frac{1}{\Gamma(1+\alpha)}\int_a^b |p(x,t)|^q (dx)^\alpha\right)^{\frac{1}{q}}$$

$$=\left[\frac{\Gamma(1+\alpha q)}{\Gamma(1+(q+1)\alpha)}\left(x^{(q+1)\alpha}-a^{(q+1)\alpha}\right)+\left(b^{(q+1)\alpha}-x^{(q+1)\alpha}\right)\right]^{\frac{1}{q}}$$

$$=\left(\frac{\Gamma(1+\alpha q)}{\Gamma(1+(q+1)\alpha)}\left(b^{(q+1)\alpha}-a^{(q+1)\alpha}\right)\right)^{\frac{1}{q}}$$

$$\leq\left(\frac{\Gamma(1+\alpha q)}{\Gamma(1+(q+1)\alpha)}\right)^{\frac{1}{q}}(b-a)^{\frac{(q+1)\alpha}{q}} \quad (12)$$

Put (7) and (12) in (8), we get

$$\left|(I-h)^\alpha f(x)+h^\alpha\frac{f(a)+f(b)}{2^\alpha}-\frac{\Gamma(1+\alpha)}{(b-a)^\alpha}aI_b^\propto f(x)\right|$$

$$\leq\frac{1}{(b-a)^\alpha}\left(\frac{\left(\Gamma(1+\alpha q)\right)}{\Gamma(1+(q+1)\alpha)}\left(b^{(q+1)\alpha}-a^{(q+1)\alpha}\right)\right)^{\frac{1}{q}}\times\left(\frac{1}{\Gamma(1+\alpha)}\right)^{\frac{1}{p}}\|f^\alpha\|_p .$$

This implies,

$$\left|(I-h)^\alpha f(x)+h^\alpha\frac{f(a)+f(b)}{2^\alpha}-\frac{\Gamma(1+\alpha)}{(b-a)^\alpha}aI_b^\propto f(x)\right|$$

$$\leq\frac{\left(\Gamma(1+\alpha q)\right)^{\frac{1}{q}}(b-a)^{\frac{\alpha}{q}}}{\left(\Gamma(1+(q+1)\alpha)\right)^{\frac{1}{q}}\left(\Gamma(1+\alpha)\right)^{\frac{1}{p}}}\|f^\alpha\|_p, 1\leq p,q\leq\infty, \frac{1}{p}+\frac{1}{q}=1$$

The first case is proved.

Case2: $0<P<1$

By using the generalized Holder's inequality described in Lemma2.2 , we obtain.

$$\frac{1}{\Gamma(1+\alpha)(b-a)^\alpha}\int_a^b |p(x,t)||f(x)^\alpha| \quad (dt)^\alpha$$

$$\leq\frac{1}{(b-a)^\alpha}\left(\frac{1}{\Gamma(1+\alpha)}\int_a^b |p(x,t)|^q (dx)^\alpha\right)^{\frac{1}{q}}\left(\frac{1}{\Gamma(1+\alpha)}\int_a^b |f(x)^\alpha|^L (dx)^\alpha\right)^{\frac{1}{L}}$$

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$$1 \le p, q \le \infty, \frac{1}{L} + \frac{1}{q} = 1$$

by using (1) and using(12) implies

$$\frac{1}{\Gamma(1+\alpha)(b-a)^{\alpha}} \int_a^b |p(x,t)||f(x)^{\alpha}| \quad (dt)^{\alpha}$$
$$\le \left(\frac{\Gamma(1+\alpha q)}{\Gamma(1+(q+1)\alpha)} \right)^{\frac{1}{q}} (b-a)^{\frac{(q+1)\alpha}{q}} \times \left(\frac{1}{\Gamma(1+\alpha)} \sum_{i=1}^{n} |f^{\alpha}(t_i)|^{L} (\Delta t_i)^{\alpha} \right)^{\frac{1}{L}}$$

Where $0 < p < 1$, by using Lemma 2.3,we get

$$\frac{1}{\Gamma(1+\alpha)(b-a)^{\alpha}} \int_a^b |p(x,t)||f(x)^{\alpha}| \quad (dt)^{\alpha}$$
$$\le \left(\frac{\Gamma(1+\alpha q)}{\Gamma(1+(q+1)\alpha)} \right)^{\frac{1}{q}} (b-a)^{\frac{(q+1)\alpha}{q}} \times \left(\frac{1}{\Gamma(1+\alpha)} \sum_{i=1}^{n} |f^{\alpha}(t_i)|^{P} (\Delta t_i)^{\alpha} \right)^{\frac{1}{P}}$$

By using (1), we get

$$\frac{1}{\Gamma(1+\alpha)(b-a)^{\alpha}} \int_a^b |p(x,t)||f(x)^{\alpha}| \quad (dt)^{\alpha}$$
$$\le \frac{C(p)\big(\Gamma(1+\alpha q)\big)^{\frac{1}{q}} (b-a)^{\frac{\alpha}{q}}}{\big(\Gamma(1+(q+1)\alpha)\big)^{\frac{1}{q}} \big(\Gamma(1+\alpha)\big)^{\frac{1}{p}}} \|f^{\alpha}\|_p, \quad 0 < p < 1.$$

Now by using Lemma 2.5, we get

$$\left| (I-h)^{\alpha} f(x) + h^{\alpha} \frac{f(a)+f(b)}{2^{\alpha}} - \frac{\Gamma(1+\alpha)}{(b-a)^{\alpha}} aI_b^{\propto} f(x) \right|$$
$$\le \frac{C(p)\big(\Gamma(1+\alpha q)\big)^{\frac{1}{q}} (b-a)^{\frac{\propto}{q}}}{\big(\Gamma(1+(q+1)\alpha)\big)^{\frac{1}{q}} \big(\Gamma(1+\alpha)\big)^{\frac{1}{p}}} \|f^{\alpha}\|_p, \qquad 0 < p < 1.$$

The second case is proved.

Corollary3.3:

If $f \subset R, f: I^{\circ} \subset R \to R^{\alpha}$ be a map $[a,b] \subset I^{\circ}$and $f \in L_{P,\alpha}[a,b]$.

Then

$$(1) \left| f(x) - \frac{\Gamma(1+\alpha)}{(b-a)^{\alpha}} aI_b^{\propto} f(x) \right| \le \frac{\big(\Gamma(1+\alpha q)\big)^{\frac{1}{q}} (b-a)^{\frac{\propto}{q}}}{\big(\Gamma(1+(q+1)\alpha)\big)^{\frac{1}{q}} \big(\Gamma(1+\alpha)\big)^{\frac{1}{p}}} \|f^{\alpha}\|_p,$$
$$1 \le p, q \le \infty, \frac{1}{p} + \frac{1}{q} = 1.$$

$$(2) \left| f(x) - \frac{\Gamma(1+\alpha)}{(b-a)^{\alpha}} aI_b^{\propto} f(x) \right|$$
$$\le \frac{C(p)\big(\Gamma(1+\alpha q)\big)^{\frac{1}{q}} (b-a)^{\frac{\propto}{q}}}{\big(\Gamma(1+(q+1)\alpha)\big)^{\frac{1}{q}} \big(\Gamma(1+\alpha)\big)^{\frac{1}{p}}} \|f^{\alpha}\|_p \quad ,0 < p < 1.$$

Proof:

By usingTheorem 3.2, we get

Case 1:

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$$\left|(I-h)^{\alpha}f(x)+h^{\alpha}\frac{f(a)+f(b)}{2^{\alpha}}-\frac{\Gamma(1+\alpha)}{(b-a)^{\alpha}}aI_b^{\propto}f(x)\right|\leq\frac{\big(\Gamma(1+\alpha q)\big)^{\frac{1}{q}}(b-a)^{\frac{\propto}{q}}}{\big(\Gamma(1+(q+1)\alpha)\big)^{\frac{1}{q}}\big(\Gamma(1+\alpha)\big)^{\frac{1}{p}}}\|f^{\alpha}\|_p,$$

If $h=0$, we get

$$\left|f(x)-\frac{\Gamma(1+\alpha)}{(b-a)^{\alpha}}aI_b^{\propto}f(x)\right|\leq\frac{\big(\Gamma(1+\alpha q)\big)^{\frac{1}{q}}(b-a)^{\frac{\propto}{q}}}{\big(\Gamma(1+(q+1)\alpha)\big)^{\frac{1}{q}}\big(\Gamma(1+\alpha)\big)^{\frac{1}{p}}}\|f^{\alpha}\|_p,\quad 1\leq p,q\leq\infty,\frac{1}{p}+\frac{1}{q}=1.$$

Case 2:

$$\left|(I-h)^{\alpha}f(x)+h^{\alpha}\frac{f(a)+f(b)}{2^{\alpha}}-\frac{\Gamma(1+\alpha)}{(b-a)^{\alpha}}aI_b^{\propto}f(x)\right|\leq\frac{C(P)\big(\Gamma(1+\alpha q)\big)^{\frac{1}{q}}(b-a)^{\frac{\propto}{q}}}{\big(\Gamma(1+(q+1)\alpha)\big)^{\frac{1}{q}}\big(\Gamma(1+\alpha)\big)^{\frac{1}{p}}}\|f^{\alpha}\|_p,$$

If $h=0$, we get

$$\left|f(x)-\frac{\Gamma(1+\alpha)}{(b-a)^{\alpha}}aI_b^{\propto}f(x)\right|\leq\frac{C(P)\big(\Gamma(1+\alpha q)\big)^{\frac{1}{q}}(b-a)^{\frac{\propto}{q}}}{\big(\Gamma(1+(q+1)\alpha)\big)^{\frac{1}{q}}\big(\Gamma(1+\alpha)\big)^{\frac{1}{p}}}\|f^{\alpha}\|_p,\quad 0<P<1.$$

Corollary 3.4:
If $f\subset R, f: I^{\circ}\subset R\to R^{\alpha}$ be a map $[a,b]\subset I^{\circ}$and $f\in L_{P,\alpha}[a,b]$.
Then

(1) $$\left|f(x)+\frac{f(a)+f(b)}{2^{\alpha}}-\frac{\Gamma(1+\alpha)}{(b-a)^{\alpha}}aI_b^{\propto}f(x)\right|$$
$$\leq\frac{\big(\Gamma(1+\alpha q)\big)^{\frac{1}{q}}(b-a)^{\frac{\propto}{q}}}{\big(\Gamma(1+(q+1)\alpha)\big)^{\frac{1}{q}}\big(\Gamma(1+\alpha)\big)^{\frac{1}{p}}}\|f^{\alpha}\|_p\quad ,1\leq p,q\leq\infty,\frac{1}{p}+\frac{1}{q}=1.$$

(2) $$\left|f(x)+\frac{f(a)+f(b)}{2^{\alpha}}-\frac{\Gamma(1+\alpha)}{(b-a)^{\alpha}}aI_b^{\propto}f(x)\right|$$
$$\leq\frac{C(p)\big(\Gamma(1+\alpha q)\big)^{\frac{1}{q}}(b-a)^{\frac{\propto}{q}}}{\big(\Gamma(1+(q+1)\alpha)\big)^{\frac{1}{q}}\big(\Gamma(1+\alpha)\big)^{\frac{1}{p}}}\|f^{\alpha}\|_p\quad ,0<p<1.$$

Proof:
By using Theorem 3.2, we get
Case 1:

$$\left|(I-h)^{\alpha}f(x)+h^{\alpha}\frac{f(a)+f(b)}{2^{\alpha}}-\frac{\Gamma(1+\alpha)}{(b-a)^{\alpha}}aI_b^{\propto}f(x)\right|\leq\frac{\big(\Gamma(1+\alpha q)\big)^{\frac{1}{q}}(b-a)^{\frac{\propto}{q}}}{\big(\Gamma(1+(q+1)\alpha)\big)^{\frac{1}{q}}\big(\Gamma(1+\alpha)\big)^{\frac{1}{p}}}\|f^{\alpha}\|_p,$$

If $h=1$, we get

$$\left|f(x)+\frac{f(a)+f(b)}{2^{\alpha}}-\frac{\Gamma(1+\alpha)}{(b-a)^{\alpha}}aI_b^{\propto}f(x)\right|\leq\frac{\big(\Gamma(1+\alpha q)\big)^{\frac{1}{q}}(b-a)^{\frac{\propto}{q}}}{\big(\Gamma(1+(q+1)\alpha)\big)^{\frac{1}{q}}\big(\Gamma(1+\alpha)\big)^{\frac{1}{p}}}\|f^{\alpha}\|_p,1\leq p,q\leq\infty,\frac{1}{p}+\frac{1}{q}=1.$$

Case 2:

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$$\left|(I-h)^{\alpha} f(x)+h^{\alpha} \frac{f(a)+f(b)}{2^{\alpha}}-\frac{\Gamma(1+\alpha)}{(b-a)^{\alpha}} aI_b^{\propto} f(x)\right| \leq \frac{C(P)\big(\Gamma(1+\alpha q)\big)^{\frac{1}{q}}(b-a)^{\frac{\propto}{q}}}{\big(\Gamma(1+(q+1)\alpha)\big)^{\frac{1}{q}}\big(\Gamma(1+\alpha)\big)^{\frac{1}{p}}}\|f^{\alpha}\|_p,$$

If $h=1$, we get

$$\left|f(x)+\frac{f(a)+f(b)}{2^{\alpha}}-\frac{\Gamma(1+\alpha)}{(b-a)^{\alpha}} aI_b^{\propto} f(x)\right|$$

$$\leq \frac{C(P)\big(\Gamma(1+\alpha q)\big)^{\frac{1}{q}}(b-a)^{\frac{\propto}{q}}}{\big(\Gamma(1+(q+1)\alpha)\big)^{\frac{1}{q}}\big(\Gamma(1+\alpha)\big)^{\frac{1}{p}}}\|f^{\alpha}\|_p \qquad 0<P<1.$$

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