

Calculation and application of the engineering numbers of Bernoulli (B_n) and Euler (E_n) to integral analysis and other applications

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Abstract: The calculation of Bernoulli and Euler numbers is based on the fact that these numbers appear in various mathematical expressions and engineering phenomena. Although computers and software programs can generate large quantities of these numbers, only a limited number are published in current references. Consequently, their use often results in unsatisfactory approximations. The objective was to calculate and evaluate the first twenty (20) Bernoulli and Euler numbers, subsequently applying them to mathematics and proposing their use in environmental engineering. There is a wide variety of relationships involving Bernoulli and Euler numbers that appear in several practical mathematical applications. In both cases, the most effective relationships used in this study are binomial expansions. By employing the appropriate and proposed equations, the Bernoulli and Euler numbers were calculated both manually and mechanically, successfully determining the number of values set as the goal. Finally, Bernoulli and Euler numbers significantly contribute to the estimation of various processes and mathematical applications, such as calculating the integrals of $\ln(\cos x)$ and $\sec(ax)/x$; respectively.

Keywords: Applications, Engineering, Bernoulli-Euler numbers, Integrals.

1. Introduction

Benton [1] notes that the Bernoulli numbers originate from a family closely associated with mathematics and engineering for several decades, even offering solutions to problems in mathematical analysis in the field of differential equations and their solution methods. Boyer [2] mentions that Euler numbers appear in the brilliant works of Leonhard Euler on *Introduction to Analysis*, *Institutiones Calculi Differentialis*, and *Algebra*.

Courant and John [3] state that the difficulty, or indeed the problem, in determining Bernoulli and Euler numbers lies in the fact that there are scientific processes and phenomena in engineering sciences where functions strongly correlate with these numbers and must be analyzed, evaluated, and interpreted carefully and meticulously. Despite frequent mentions of the extensive range of Bernoulli and Euler numbers, only a limited subset is published, resulting in the use of alternative solutions such as approximation techniques. In this sense, their determination is well justified. Benton [1] reports that

Jacob Bernoulli published his book *Ars Conjectandi*, where, for the first and only time, a large set of mathematical expressions and relationships was presented, defined as important and suitable formulas for calculating the sums of positive integer powers of the first n natural numbers Abramowitz and Stegun [4] further states that all Bernoulli numbers are positive and, from the fourth onward, grow without bound. This growth can be expressed by the relationship $\lim_{m \rightarrow \infty} \left(\frac{B_m}{B_{m-1}} \right) = \frac{m^2}{\pi^2}$. According to Fernández [5] and Vera [6], the Bernoulli were first used in numerous practical cases by Moivre and Euler. Bronshtein and Semendiaev [7] indicate that Euler numbers are quantitatively considered and generally related through simple and complex relationships, equations, and functions within integral and differential calculus, and in phenomena that interconnect these mathematical concepts. Bronshtein and Semendiaev [7] and Marín [8] mention that both Bernoulli and Euler numbers have practical applications in applied and abstract mathematical sciences across various fields of knowledge and in some engineering science applications. One of the eleven descendants and disciples of the famous Bernoulli family moved to Basel, Switzerland, to continue mathematical training and preserve the legacy of the Bernoulli lineage. Child [9] and Fernández [5] mention that, due to the great importance and versatility of Bernoulli numbers, they were published for the first time in 1713, in the splendid, magnificent, and excellent work titled *Ars Conjectandi*, after a period of eight years had passed since the death of its great author, Jacob Bernoulli.

Ruíz [10] and Martínez et al. [11] indicate, after a rigorous mathematical analysis, that with respect to Bernoulli numbers, the frequent and accurate relationships, in the form of series, are those that contain these everyday, immeasurable, and greatly important numbers. With respect to Bernoulli numbers, the series-form relationships are those that contain these numbers, Martínez et al. [11] and Ruíz [10]. Vega-Calderón et al. [12] indicate that Bernoulli numbers have no direct relationship with Bernoulli's equation, and that this equation, regarding fluid flow, presents certain difficulties for students of subjects that describe fluid flow phenomena or behaviors of water properties. It is also mentioned that Bernoulli numbers were published and initially corroborated as validation and proof in the year 1713, in the work titled *Ars Conjectandi*, ten years after the death of its progenitor and author, Fernández [5] and Child [9].

Vega-Calderón et al. [12] mention, after extensive and rigorous discussion and analysis, that Bernoulli numbers have no direct or indirect relationship with Bernoulli's equation, widely applied and used in the treatment of phenomena and processes involving the flow of substances with thermophysical properties analogous to water, and the different relationships with said equation, regarding fluid flow, present certain difficulties to be understood and applied by students of subjects dealing with runoff processes and describing fluid flow phenomena, whatever these may be.

Sierra Cotes and Gómez Yance [13] indicate that, when studying the behavior of diameter, jet velocity, and flow with different vertical discharge heights, the process followed the trends of Bernoulli number functions, which consisted of the study of flow through an orifice, where the velocity of the fluid is studied analogous to what the Bernoulli numbers and equation describe for the outflow of regulatory boxes.

Huancapaza [14] indicates that his objective was to determine which numerical method is most effective for obtaining the dynamic response of multi-degree-of-freedom structures, resulting in Euler numbers describing a Gaussian behavior with increasing time.

Torres [15] states in his research, "Generation of a mathematical model for the study of reservoir operation with a multipurpose approach," where the objective was the development of a mathematical model using the principles of normal distribution mechanics to simulate reservoir operations, resulting in Euler numbers and their method being the simplest numerical scheme.

Zamora et al. [16] argue that Bernoulli numbers represent asymptotic decay processes regarding the conservation of mechanical energy, allowing analytical and experimental methods to determine different parameters such as tank emptying time, pipe area, flow velocity, and velocity head. Estela Urbina et al. [17] indicate that Bernoulli numbers are very important in asymptotic decay processes,

being an adequate strategy to solve problematic situations of complex short transitions, considering that friction or drag losses are negligible.

Estela Urbina et al. [17] affirm that Bernoulli numbers are very important in asymptotic decay processes, being an adequate strategy to solve problematic situations of complex short transitions, considering that friction or drag losses are negligible.

Marín-Machuca et al. [18] mention in their research "Quantification of the alkalinity of natural water applying Bernoulli numbers and Simpson's approximate integral" that the purpose was to calculate the first twenty Bernoulli numbers and apply them to mathematics ($\int \ln(\cos x) dx$) and to environmental engineering, quantifying the alkalinity of natural water in the form of calcium carbonate ($(C_aCO_3)(mg/l)$), measuring the carbon dioxide (CO_2); that there exists, with certainty and specifically, a great variety of mathematical relationships and expressions that include Bernoulli numbers, appearing in various practical applications of mathematics; that the best relationships used on this occasion are binomial expansions; that by applying the proposed and suitable equations, Bernoulli numbers were satisfactorily calculated, reaching the number of values set as the objective, and finally, Bernoulli numbers decisively and significantly contribute to mathematical processes such as calculating integrals of the type $\ln(\cos x)$, as well as applications in environmental engineering, evaluating water alkalinity as calcium carbonate (C_aCO_3).

Child [9] mention that the anecdote told by Gauss (1777–1855) about his early years in primary school is well known, when he impressed his bad-tempered mathematics teacher with a very simple but ingenious procedure for summing the first 100 terms of an arithmetic progression; however, it is not as well known that a hundred years earlier, one of the most outstanding mathematicians of the Bernoulli family, Bernoulli [19] left for posterity an excellent, outstanding, and brilliant work, which could well be considered the first great treatise on combinatorics and probability: Bernoulli [19] where in this brilliant work, highly recommended reading, there appears a result far superior to Gauss's sum, asking: What do you think if we calculate the sum of the first 1000 powers of 10 of the natural numbers? But of course, with pencil and paper, or quill, which were the tools our dear friend, mathematician, and teacher Bernoulli had at that time, kindly presenting in this article some of the essential and fundamental ideas that led him to carry out such purpose and which were also the origin, beginning, and foundation of some numbers that today bear his name: "the Bernoulli numbers."

Martínez et al. [11] have developed the main and relevant aspects regarding Bernoulli polynomials and numbers, which serve, due to their importance and foundation, for the presentation of the results found to test, conjecture, and prove the Faulhaber Formula Theorem, in which the cotangent function is developed in a series, and to infinity, in relation to Bernoulli numbers and in this way the values of the Riemann zeta function are evaluated at integer arguments through Bernoulli numbers, and finally the explicit formulas for evaluating and calculating Bernoulli numbers in terms of Stirling numbers of the first and second kind are verified.

The purpose has been to determine and evaluate the first twenty Bernoulli and Euler numbers, to present each of them in table form, and to apply them, in some way, in the calculation of integrals with expressions containing said numbers.

$$\frac{x}{2} \cot \frac{x}{2} = 1 - B_1 \frac{x^2}{2!} - B_2 \frac{x^4}{4!} - B_3 \frac{x^6}{6!} - \dots$$

$$\frac{x e^{x+1}}{2 e^{x-1}} = 1 + B_1 \frac{x^2}{2!} - B_2 \frac{x^4}{4!} + B_3 \frac{x^6}{6!} - \dots$$

$$B_m = \frac{(2m)!}{2^{2m-1} \pi^{2m}} \left[1 + \frac{1}{2^{2m}} + \frac{1}{3^{2m}} + \frac{1}{4^{2m}} + \dots \right]$$

$$B_m = 4m \int \frac{t^{2m-1} dt}{e^{2\pi t} - 1}$$

$$tg.x = 2^2(2^2 - 1)B_1 \frac{x}{2!} + 2^4(2^4 - 1)B_2 \frac{x^3}{4!} + \dots$$

$$\log \frac{\sin x}{x} = -\frac{2}{1} B_1 \frac{x^2}{2!} - \frac{2^3}{2} B_2 \frac{x^4}{4!} - \frac{2^5}{3} B_3 \frac{x^6}{6!} - \dots$$

$$\frac{x}{e^x-1} = \sum_{n=0}^{\infty} \frac{B_n x^n}{n!} = B_0 + B_1 x + \frac{B_2}{2!} x^2 + \frac{B_3}{3!} x^3 + \dots$$

$$\frac{x}{e^x-1} = 1 - \frac{x}{2} + \frac{B_1 x^2}{2!} - \frac{B_2 x^4}{4!} + \frac{B_3 x^6}{6!} - \dots$$

$$x \cdot \cot x = \sum_{n=0}^{\infty} \frac{(-1)^n B_{2n} (2x)^{2n}}{(2n)!}$$

$$1 - \frac{x}{2} \cot \frac{x}{2} = \frac{B_1 x^2}{2!} + \frac{B_2 x^4}{4!} + \frac{B_3 x^6}{6!} + \dots$$

$$B_{2n} = \frac{2(-1)^{n-1}(2n)!}{(2\pi)^{2n}} \left(1 + \frac{1}{2^{2n}} + \frac{1}{3^{2n}} + \frac{1}{4^{2n}} + \dots \right)$$

$$\binom{2n+1}{2} 2^2 B_1 - \binom{2n+1}{4} 2^4 B_2 + \binom{2n+1}{6} 2^6 B_3 - \dots - (-1)^{n-1} (2n+1) 2^{2n} B_n = 2n$$

Bronshtein and Semendiaev [7] mention the other series that relate to the Bernoulli numbers:

$$B_n = \frac{2(2n)!}{(2^{2n-1})\pi^{2n}} \left(1 + \frac{1}{3^{2n}} + \frac{1}{5^{2n}} + \frac{1}{7^{2n}} + \dots \right)$$

$$B_n = \frac{(2n)!}{(2^{2n-1}-1)\pi^{2n}} \left(1 - \frac{1}{2^{2n}} + \frac{1}{3^{2n}} - \frac{1}{4^{2n}} + \dots \right)$$

$$B_n = \frac{(2n)!}{2^{2n-1}\pi^{2n}} \left(1 + \frac{1}{2^{2n}} + \frac{1}{3^{2n}} + \frac{1}{4^{2n}} + \frac{1}{5^{2n}} + \dots \right)$$

$$B_n = n! \times \begin{vmatrix} \frac{1}{2!} & 1 & 0 & \dots & \dots & \dots & 0 \\ \frac{1}{3!} & \frac{1}{2!} & 1 & \dots & \dots & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{1}{(n+1)!} & \frac{1}{n!} & \frac{1}{(n-1)!} & \dots & \dots & \dots & \frac{1}{2!} \end{vmatrix}$$

Fernández [5], Varadarajan [20] and Bronshtein and Semendiaev [7] indicate that the expressions relating to Euler numbers are:

$$1 - \frac{1}{3^{2n+1}} + \frac{1}{5^{2n+1}} - \frac{1}{7^{2n+1}} + \dots \pm \frac{1}{(2n-1)^{2n+1}} \mp \dots = \frac{\pi^{2n+1}}{2^{2n+2} \times (2n)!} \times E_n$$

$$\sec x = 1 + \frac{E_1 x^2}{2!} + \frac{E_2 x^4}{4!} + \frac{E_3 x^6}{6!} + \dots$$

$$\sec h x = 1 - \frac{E_1 x^2}{2!} + \frac{E_2 x^4}{4!} - \frac{E_3 x^6}{6!} + \dots$$

$$E_n = \frac{2^{2n+2}(2n)!}{\pi^{2n+1}} \left(1 - \frac{1}{3^{2n+1}} + \frac{1}{5^{2n+1}} - \frac{1}{7^{2n+1}} + \frac{1}{9^{2n+1}} - \dots \right)$$

$$E_n = \begin{vmatrix} \frac{1}{2!} & 1 & \dots & 0 & 0 \\ \frac{1}{4!} & \frac{1}{2!} & \dots & 0 & 0 \\ \vdots & \vdots & \dots & \vdots & \vdots \\ \vdots & \vdots & \dots & \vdots & \vdots \\ \vdots & \vdots & \dots & 1 & 0 \\ \vdots & \vdots & \dots & \frac{1}{2!} & 1 \\ \frac{1}{2n!} & \frac{1}{2n-2!} & \dots & \frac{1}{4!} & \frac{1}{2!} \end{vmatrix}$$

$$E_n = \begin{vmatrix} 1 & 1 & 0 & \dots & 0 \\ 1 & \binom{4}{2} & 1 & \dots & 0 \\ 1 & \binom{6}{2} & \binom{6}{4} & \dots & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 1 & \binom{2n-2}{2} & \binom{2n-2}{4} & \dots & 1 \\ 1 & \binom{2n}{2} & \binom{2n}{4} & \dots & \binom{2n}{2n-2} \end{vmatrix}$$

2. Materials y Methods

Methodology. The research has been divided into two parts. In the first part, we focused on calculating the first twenty Bernoulli numbers, and in the second part, we focused on calculating the first twenty Euler numbers. The most convenient, suitable, and important expression that relates Bernoulli numbers, according to Tsuneo et al. [21] and Fernández [5], which has been used in this study, is:

$$\binom{2n+1}{1} B_n - \binom{2n+1}{3} B_{n-1} + \binom{2n+1}{5} B_{n-2} + (-1)^{n-1} \binom{2n+1}{2n-1} B_1 + (-1)^n \left(n - \frac{1}{2}\right) = 0,$$

Procedure: Calculation of Bernoulli Numbers

$$\binom{2n+1}{1} B_n - \binom{2n+1}{3} B_{n-1} + \binom{2n+1}{5} B_{n-2} + (-1)^{n-1} \binom{2n+1}{2n-1} B_1 + (-1)^n \left(n - \frac{1}{2}\right) = 0,$$

The above expression has been used to evaluate the Bernoulli numbers (B_n); whose calculations are presented according to the development outlined in summarized form below:

Calculation, according to the binomial formula for the Bernoulli number, of B_1

$$\binom{3}{1} B_1 - \left(1 - \frac{1}{2}\right) = 0; \frac{3!}{1! \times 2!} B_1 = \frac{1}{2} \Rightarrow B_1 = \frac{1}{6}$$

Calculation, according to the binomial formula for the Bernoulli number, of B_2

$$\binom{5}{1} B_2 - \binom{5}{3} \times \frac{1}{6} + \left(2 - \frac{1}{2}\right) = 0; \frac{5!}{1! \times 4!} B_2 - \frac{5!}{3! \times 2!} \times \frac{1}{6} + \frac{3}{2} = 0$$

$$\Rightarrow B_2 = \frac{1}{30}$$

Calculation, according to the binomial formula for the Bernoulli number, of B_3

$$\binom{7}{1} \times B_3 - \binom{7}{3} \times \frac{1}{30} + \binom{7}{5} \times \frac{1}{6} - \left(3 - \frac{1}{2}\right) = 0; \frac{7!}{1! \times 6!} \times B_3 - \frac{7!}{3! \times 4!} \times \frac{1}{30} + \frac{7!}{5! \times 2!} \times \frac{1}{6} - \frac{5}{2} = 0$$

$$\Rightarrow B_3 = \frac{1}{42}$$

Calculation, according to the binomial formula for the Bernoulli number, of B_4

$$\binom{9}{1} \times B_4 - \binom{9}{3} \times \frac{1}{42} + \binom{9}{5} \times \frac{1}{30} - \binom{9}{7} \times \frac{1}{6} + \left(4 - \frac{1}{2}\right) = 0$$

$$\frac{9!}{1! \times 8!} \times B_4 - \frac{9!}{3! \times 6!} \times \frac{1}{42} + \frac{9!}{5! \times 4!} \times \frac{1}{30} - \frac{9!}{7! \times 2!} \times \frac{1}{6} + \frac{7}{2} = 0$$

$$\Rightarrow B_4 = \frac{1}{30}$$

Calculation, according to the binomial formula for the Bernoulli number, of B_5

$$\begin{aligned} \binom{11}{1} B_5 - \binom{11}{3} \frac{1}{30} + \binom{11}{5} \frac{1}{42} - \binom{11}{7} \frac{1}{30} + \binom{11}{9} \frac{1}{6} - \left(5 - \frac{1}{2}\right) &= 0 \\ \frac{11!}{1! \times 10!} B_5 - \frac{11!}{3! \times 8!} \frac{1}{30} + \frac{11!}{5! \times 6!} \times \frac{1}{42} - \frac{11!}{7! \times 4!} \frac{1}{30} + \frac{11!}{9! \times 2!} \frac{1}{6} - \frac{9}{2} &= 0 \\ \Rightarrow B_5 &= \frac{5}{66} \end{aligned}$$

Calculation, according to the binomial formula for the Bernoulli number, of B_6

$$\begin{aligned} \binom{13}{1} B_6 - \binom{13}{3} \frac{5}{66} + \binom{13}{5} \frac{1}{30} - \binom{13}{7} \frac{1}{42} + \binom{13}{9} \frac{1}{30} - \binom{13}{11} \frac{1}{6} + \left(6 - \frac{1}{2}\right) &= 0 \\ \frac{13!}{1! \times 12!} B_6 - \frac{13!}{3! \times 10!} \frac{5}{66} + \frac{13!}{5! \times 8!} \frac{1}{30} - \frac{13!}{7! \times 6!} \frac{1}{42} + \frac{13!}{9! \times 4!} \frac{1}{30} - \frac{13!}{11! \times 2!} \frac{1}{6} + \frac{11}{2} &= 0 \\ \Rightarrow B_6 &= \frac{691}{2730} \end{aligned}$$

Calculation, according to the binomial formula for the Bernoulli number, of B_7

$$\begin{aligned} \binom{15}{1} B_7 - \binom{15}{3} \frac{691}{2730} + \binom{15}{5} \frac{5}{66} - \binom{15}{7} \frac{1}{30} + \binom{15}{9} \frac{1}{42} - \binom{15}{11} \frac{1}{30} + \binom{15}{13} \frac{1}{6} - \left(7 - \frac{1}{2}\right) &= 0 \\ \frac{15!}{1! \times 14!} B_7 - \frac{15!}{3! \times 12!} \frac{691}{2730} + \frac{15!}{5! \times 10!} \frac{5}{66} - \frac{15!}{7! \times 8!} \frac{1}{30} + \frac{15!}{9! \times 6!} \frac{1}{42} - \frac{15!}{11! \times 4!} \frac{1}{30} + \frac{15!}{13! \times 2!} \frac{1}{6} - \frac{13}{2} &= 0 \\ \Rightarrow B_7 &= \frac{7}{6} \end{aligned}$$

Calculation, according to the binomial formula for the Bernoulli number, of B_8

$$\begin{aligned} \binom{17}{1} B_8 - \binom{17}{3} \frac{7}{6} + \binom{17}{5} \frac{691}{2730} - \binom{17}{7} \frac{5}{66} + \binom{17}{9} \frac{1}{30} - \binom{17}{11} \frac{1}{42} + \binom{17}{13} \frac{1}{30} - \binom{17}{15} \frac{1}{6} + \left(8 - \frac{1}{2}\right) &= 0 \\ \frac{17!}{1! \times 16!} B_8 - \frac{17!}{3! \times 14!} \frac{7}{6} + \frac{17!}{5! \times 12!} \frac{691}{2730} - \frac{17!}{7! \times 10!} \frac{5}{66} + \frac{17!}{9! \times 8!} \frac{1}{30} - \frac{17!}{11! \times 6!} \frac{1}{42} + \frac{17!}{13! \times 4!} \frac{1}{30} - \frac{17!}{15! \times 2!} \frac{1}{6} + \frac{15}{2} &= 0 \\ \Rightarrow B_8 &= \frac{3617}{510} \end{aligned}$$

Calculation, according to the binomial formula for the Bernoulli number, of B_9

$$\begin{aligned} \binom{19}{1} B_9 - \binom{19}{3} \frac{3617}{510} + \binom{19}{5} \frac{7}{6} - \binom{19}{7} \frac{691}{2730} + \binom{19}{9} \frac{5}{66} - \binom{19}{11} \frac{1}{30} + \binom{19}{13} \frac{1}{42} - \binom{19}{15} \frac{1}{30} + \binom{19}{17} \frac{1}{6} - \left(9 - \frac{1}{2}\right) &= 0 \\ \frac{19!}{1! \times 18!} B_9 - \frac{19!}{3! \times 16!} \frac{3617}{510} + \frac{19!}{5! \times 14!} \frac{7}{6} - \frac{19!}{7! \times 12!} \frac{691}{2730} + \frac{19!}{9! \times 10!} \frac{5}{66} - \frac{19!}{11! \times 8!} \frac{1}{30} + \frac{19!}{13! \times 6!} \frac{1}{42} - \frac{19!}{15! \times 4!} \frac{1}{30} + \frac{19!}{17! \times 2!} \frac{1}{6} - \frac{17}{2} &= 0 \\ \Rightarrow B_9 &= \frac{43867}{798} \end{aligned}$$

Calculation, according to the binomial formula for the Bernoulli number, of B_{10}

$$\begin{aligned} \binom{21}{1} B_{10} - \binom{21}{3} \frac{43867}{798} + \binom{21}{5} \frac{3617}{510} - \binom{21}{7} \frac{7}{6} + \binom{21}{9} \frac{691}{2730} - \binom{21}{11} \frac{5}{66} + \binom{21}{13} \frac{1}{30} - \binom{21}{15} \frac{1}{42} + \binom{21}{17} \frac{1}{6} - \left(10 - \frac{1}{2}\right) &= 0 \\ \frac{21!}{1! \times 20!} B_{10} - \frac{21!}{3! \times 18!} \frac{43867}{798} + \frac{21!}{5! \times 16!} \frac{3617}{510} - \frac{21!}{7! \times 14!} \frac{7}{6} + \frac{21!}{9! \times 12!} \frac{691}{2730} - \frac{21!}{11! \times 10!} \frac{5}{66} + \frac{21!}{13! \times 8!} \frac{1}{30} - \frac{21!}{15! \times 6!} \frac{1}{42} + \frac{21!}{17! \times 4!} \frac{1}{6} - \frac{19}{2} &= 0 \\ \Rightarrow B_{10} &= \frac{174611}{330} \end{aligned}$$

Calculation, according to the binomial formula for the Bernoulli number, of B_{11}

$$\begin{aligned} \binom{23}{1} B_{11} - \binom{23}{3} \frac{174611}{330} + \binom{23}{5} \frac{43867}{798} - \binom{23}{7} \frac{3617}{510} + \binom{23}{9} \frac{7}{6} - \binom{23}{11} \frac{691}{2730} + \binom{23}{13} \frac{5}{66} - \binom{23}{15} \frac{1}{30} + \binom{23}{17} \frac{1}{42} - \binom{23}{19} \frac{1}{30} + \binom{23}{21} \frac{1}{6} - \left(11 - \frac{1}{2}\right) &= 0 \end{aligned}$$

$$\frac{23!}{1! \times 22!} B_{11} - \frac{23!}{3! \times 20!} \frac{174611}{330} + \frac{23!}{5! \times 18!} \frac{43867}{798} - \frac{23!}{7! \times 16!} \frac{3617}{510} + \frac{23!}{9! \times 14!} \frac{7}{6} - \frac{23!}{11! \times 12!} \frac{691}{2730} + \frac{23!}{13! \times 10!} \frac{5}{66} - \frac{23!}{15! \times 8!} \frac{1}{30} + \frac{23!}{17! \times 6!} \frac{1}{42} - \frac{23!}{19! \times 4!} \frac{1}{30} + \frac{23!}{21! \times 2!} \frac{1}{6} - \frac{21}{2} = 0$$

$$\Rightarrow B_{11} = \frac{854513}{138}$$

Calculation, according to the binomial formula for the Bernoulli number, of B_{12}

$$\begin{aligned} & \binom{25}{1} B_{12} - \binom{25}{3} \frac{854513}{138} + \binom{25}{5} \frac{174611}{330} - \binom{25}{7} \frac{43867}{798} + \binom{25}{9} \frac{3617}{510} - \binom{25}{11} \frac{7}{6} + \\ & + \binom{25}{13} \frac{691}{2730} - \binom{25}{15} \frac{5}{66} + \binom{25}{17} \frac{1}{30} - \binom{25}{19} \frac{1}{42} + \binom{25}{21} \frac{1}{30} - \binom{25}{23} \frac{1}{6} + \left(12 - \frac{1}{2}\right) = 0 \\ & \frac{25!}{1! \times 24!} B_{12} - \frac{25!}{3! \times 22!} \frac{138}{25!} + \frac{25!}{5! \times 20!} \frac{330}{25!} - \frac{25!}{7! \times 18!} \frac{798}{25!} + \frac{25!}{9! \times 16!} \frac{510}{25!} - \frac{25!}{11! \times 14!} \frac{6}{25!} + \frac{25!}{13! \times 12!} \frac{691}{2730} - \\ & - \frac{25!}{15! \times 10!} \frac{66}{25!} + \frac{25!}{17! \times 8!} \frac{30}{25!} - \frac{25!}{19! \times 6!} \frac{42}{25!} + \frac{25!}{21! \times 4!} \frac{30}{25!} - \frac{25!}{23! \times 2!} \frac{6}{25!} + \frac{23}{2} = 0 \\ & \Rightarrow B_{12} = \frac{236364091}{2730} \end{aligned}$$

Calculation, according to the binomial formula for the Bernoulli number, of B_{13}

$$\begin{aligned} & \binom{27}{1} B_{13} - \binom{27}{3} \frac{236364091}{2730} + \binom{27}{5} \frac{854513}{138} - \binom{27}{7} \frac{174611}{330} + \binom{27}{9} \frac{43867}{798} - \binom{27}{11} \frac{3617}{510} + \binom{27}{13} \frac{7}{6} - \\ & - \binom{27}{15} \frac{691}{2730} + \binom{27}{17} \frac{5}{66} - \binom{27}{19} \frac{1}{30} + \binom{27}{21} \frac{1}{42} - \binom{27}{23} \frac{1}{30} + \binom{27}{25} \frac{1}{6} - \left(13 - \frac{1}{2}\right) = 0 \\ & \frac{27!}{1! \times 26!} B_{13} - \frac{27!}{3! \times 24!} \frac{2730}{27!} + \frac{27!}{5! \times 22!} \frac{138}{27!} - \frac{27!}{7! \times 20!} \frac{330}{27!} + \frac{27!}{9! \times 18!} \frac{798}{27!} - \frac{27!}{11! \times 16!} \frac{510}{27!} + \frac{27!}{13! \times 14!} \frac{6}{27!} - \\ & - \frac{27!}{15! \times 12!} \frac{691}{2730} + \frac{27!}{17! \times 10!} \frac{66}{27!} - \frac{27!}{19! \times 8!} \frac{30}{27!} + \frac{27!}{21! \times 6!} \frac{42}{27!} - \frac{27!}{23! \times 4!} \frac{30}{27!} + \frac{27!}{25! \times 2!} \frac{6}{27!} - \frac{25}{2} = 0 \\ & \Rightarrow B_{13} = \frac{8553103}{6} \end{aligned}$$

Calculation, according to the binomial formula for the Bernoulli number, of B_{14}

$$\begin{aligned} & \binom{29}{1} B_{14} - \binom{29}{3} \frac{8553103}{6} + \binom{29}{5} \frac{236364091}{2730} - \binom{29}{7} \frac{854513}{138} + \binom{29}{9} \frac{174611}{330} - \binom{29}{11} \frac{43867}{798} + \\ & + \binom{29}{13} \frac{3617}{510} - \binom{29}{15} \frac{7}{6} + \binom{29}{17} \frac{691}{2730} - \binom{29}{19} \frac{5}{66} + \binom{29}{21} \frac{1}{30} - \binom{29}{23} \frac{1}{42} + \binom{29}{25} \frac{1}{30} - \binom{29}{27} \frac{1}{6} + \left(14 - \frac{1}{2}\right) = 0 \\ & \frac{29!}{1! \times 28!} B_{14} - \frac{29!}{3! \times 26!} \frac{6}{29!} + \frac{29!}{5! \times 24!} \frac{2730}{29!} - \frac{29!}{7! \times 22!} \frac{138}{29!} + \frac{29!}{9! \times 20!} \frac{330}{29!} - \frac{29!}{11! \times 18!} \frac{798}{29!} + \frac{29!}{13! \times 16!} \frac{510}{29!} - \\ & - \frac{29!}{15! \times 14!} \frac{6}{29!} + \frac{29!}{17! \times 12!} \frac{2730}{29!} - \frac{29!}{19! \times 10!} \frac{66}{29!} + \frac{29!}{21! \times 8!} \frac{30}{29!} - \frac{29!}{23! \times 6!} \frac{42}{29!} + \frac{29!}{25! \times 4!} \frac{30}{29!} - \frac{29!}{27! \times 2!} \frac{6}{29!} + \frac{27}{2} = 0 \\ & \Rightarrow B_{14} = \frac{23749461029}{870} \end{aligned}$$

Calculation, according to the binomial formula for the Bernoulli number, of B_{15}

$$\begin{aligned} & \binom{31}{1} B_{15} - \binom{31}{3} \frac{23749461029}{870} + \binom{31}{5} \frac{8553103}{6} - \binom{31}{7} \frac{236364091}{2730} + \binom{31}{9} \frac{854513}{138} - \binom{31}{11} \frac{174611}{330} + \\ & + \binom{31}{13} \frac{43867}{798} - \binom{31}{15} \frac{3617}{510} + \binom{31}{17} \frac{7}{6} - \binom{31}{19} \frac{691}{2730} + \binom{31}{21} \frac{5}{66} - \binom{31}{23} \frac{1}{30} + \binom{31}{25} \frac{1}{42} - \binom{31}{27} \frac{1}{30} + \binom{31}{29} \frac{1}{6} - \\ & - \left(15 - \frac{1}{2}\right) = 0 \\ & \frac{31!}{1! \times 30!} B_{15} - \frac{31!}{3! \times 28!} \frac{23749461029}{31!} + \frac{31!}{5! \times 26!} \frac{870}{31!} - \frac{31!}{7! \times 24!} \frac{2730}{31!} + \frac{31!}{9! \times 22!} \frac{138}{31!} - \frac{31!}{11! \times 20!} \frac{330}{31!} + \\ & + \frac{31!}{13! \times 18!} \frac{798}{31!} - \frac{31!}{15! \times 16!} \frac{510}{31!} + \frac{31!}{17! \times 14!} \frac{6}{31!} - \frac{31!}{19! \times 12!} \frac{2730}{31!} + \frac{31!}{21! \times 10!} \frac{66}{31!} - \frac{31!}{23! \times 8!} \frac{30}{31!} + \frac{31!}{25! \times 6!} \frac{42}{31!} - \frac{31!}{27! \times 4!} \frac{30}{31!} + \\ & + \frac{31!}{29! \times 2!} \frac{6}{31!} - \frac{29}{2} = 0 \\ & \Rightarrow B_{15} = \frac{8615841276005}{14322} \end{aligned}$$

With respect to the Euler numbers, the most important, convenient, and appropriate expression that relates Euler numbers according to Varadarajan [20] and Murray [22] and which was used to calculate the Euler numbers on this occasion, is:

$$E_n = \binom{2n}{2} \times E_{n-1} - \binom{2n}{4} \times E_{n-2} + \binom{2n}{6} \times E_{n-3} - \binom{2n}{8} E_{n-4} + \dots (-1)^n$$

The above expression has been used to evaluate the Euler numbers (E_n); whose calculations are presented in the development detailed below:

Calculation, according to the binomial formula for the Euler number, of E_1

$$E_1 = -(-1)^1 \Rightarrow E_1 = 1$$

Calculation, according to the binomial formula for the Euler number, of E_2

$$E_2 = \binom{4}{2} \times 1 - (-1)^2 = \frac{4!}{2! \times 2!} \times 1 - 1 \Rightarrow E_2 = 5$$

Calculation, according to the binomial formula for the Euler number, of E_3

$$E_3 = \binom{6}{2} \times 5 - \binom{6}{4} \times 1 - (-1)^3 = \frac{6!}{4! \times 2!} \times 5 - \frac{6!}{2! \times 4!} \times 1 + 1 \Rightarrow E_3 = 61$$

Calculation, according to the binomial formula for the Euler number, of E_4

$$E_4 = \binom{8}{2} 61 - \binom{8}{4} 5 + \binom{8}{6} 1 - (-1)^4; E_4 = \frac{8!}{6! \times 2!} 61 - \frac{8!}{4! \times 4!} 5 + \frac{8!}{2! \times 6!} 1 - 1 \Rightarrow E_4 = 1385$$

Calculation, according to the binomial formula for the Euler number, of E_5

$$E_5 = \binom{10}{2} 1385 - \binom{10}{4} 61 + \binom{10}{6} 5 - \binom{10}{8} 1 + (-1)^5$$

$$E_5 = \frac{10!}{8! \times 2!} 1385 - \frac{10!}{6! \times 4!} 61 + \frac{10!}{4! \times 6!} 5 - \frac{10!}{2! \times 8!} 1 + 1$$

$$E_5 = 50521$$

Calculation, according to the binomial formula for the Euler number, of E_6

$$E_6 = \binom{12}{2} 50521 - \binom{12}{4} 1385 + \binom{12}{6} 61 - \binom{12}{8} 5 + \binom{12}{10} 1 - (-1)^6$$

$$E_6 = \frac{12!}{10! \times 2!} 50521 - \frac{12!}{8! \times 4!} 1385 + \frac{12!}{6! \times 6!} 61 - \frac{12!}{4! \times 8!} 5 + \frac{12!}{2! \times 10!} 1 - 1$$

$$E_6 = 2702765$$

Calculation, according to the binomial formula for the Euler number, of E_7

$$E_7 = \binom{14}{2} 2702765 - \binom{14}{4} 50521 + \binom{14}{6} 1385 - \binom{14}{8} 61 + \binom{14}{10} 5 - \binom{14}{12} 1 - (-1)^7$$

$$E_7 = \frac{14!}{12! \times 2!} 2702765 - \frac{14!}{10! \times 4!} 50521 + \frac{14!}{8! \times 6!} 1385 - \frac{14!}{6! \times 8!} 61 + \frac{14!}{4! \times 10!} 5 - \frac{14!}{2! \times 12!} 1 + 1$$

$$\Rightarrow E_7 = 199360981$$

Calculation, according to the binomial formula for the Euler number, of E_8

$$E_8 = \binom{16}{2} 199360981 - \binom{16}{4} 2702765 + \binom{16}{6} 50521 - \binom{16}{8} 1385 + \binom{16}{10} 61 - \binom{16}{12} 5 + \binom{16}{14} 1 - (-1)^8$$

$$E_8 = \frac{16!}{14! \times 2!} 199360981 - \frac{16!}{12! \times 4!} 2702765 + \frac{16!}{10! \times 6!} 50521 - \frac{16!}{8! \times 8!} 1385 + \frac{16!}{6! \times 10!} 61 - \frac{16!}{4! \times 12!} 5 + \frac{16!}{2! \times 14!} 1 - 1$$

$$\Rightarrow E_8 = 19391512145$$

Calculation, according to the binomial formula for the Euler number, of E_9

$$E_9 = \binom{18}{2} 19391512145 - \binom{18}{4} 199360981 + \binom{18}{6} 2702765 - \binom{18}{8} 50521 + \binom{18}{10} 1385 - \binom{18}{12} 61 + \binom{18}{14} 5 - \binom{18}{16} 1 - (-1)^9$$

$$E_9 = \frac{18!}{16! \times 2!} 19391512145 - \frac{18!}{14! \times 4!} 199360981 + \frac{18!}{12! \times 6!} 2702765 - \frac{18!}{10! \times 8!} 50521 + \frac{18!}{8! \times 10!} 1385 - \frac{18!}{6! \times 12!} 61 + \frac{18!}{4! \times 14!} 5 - \frac{18!}{2! \times 16!} 1 + 1$$

$$\Rightarrow E_9 = 2404879675441$$

Calculation, according to the binomial formula for the Euler number, of E_{10}

$$E_{10} = \binom{20}{2} 2404879675441 - \binom{20}{4} 19391512145 + \binom{20}{6} 199360981 - \binom{20}{8} 2702765 + \binom{20}{10} 50521 - \binom{20}{12} 1385 + \binom{20}{14} 61 - \binom{20}{16} 5 + \binom{20}{18} 1 - (-1)^{10}$$

$$E_{10} = \frac{20!}{18! \times 2!} 2404879675441 - \frac{20!}{16! \times 4!} 19391512145 + \frac{20!}{14! \times 6!} 199360981 - \frac{20!}{12! \times 8!} 2702765 + \frac{20!}{10! \times 10!} 50521 - \frac{20!}{8! \times 12!} 1385 + \frac{20!}{6! \times 14!} 61 - \frac{20!}{4! \times 16!} 5 + \frac{20!}{2! \times 18!} 1 - 1$$

$$\Rightarrow E_{10} = 370371188237525$$

Calculation, according to the binomial formula for the Euler number, of E_{11}

$$E_{11} = \binom{22}{2} 370371188237525 - \binom{22}{4} 2404879675441 + \binom{22}{6} 19391512145 - \binom{22}{8} 199360981 + \binom{22}{10} 2702765 - \binom{22}{12} 50521 + \binom{22}{14} 1385 - \binom{22}{16} 61 + \binom{22}{18} 5 - \binom{22}{20} 1 - (-1)^{11}$$

$$E_{11} = \frac{22!}{20! \times 2!} 370371188237525 - \frac{22!}{18! \times 4!} 2404879675441 + \frac{22!}{16! \times 6!} 19391512145 - \frac{22!}{14! \times 8!} 199360981 + \frac{22!}{12! \times 10!} 2702765 - \frac{22!}{10! \times 12!} 50521 + \frac{22!}{8! \times 14!} 1385 - \frac{22!}{6! \times 16!} 61 + \frac{22!}{4! \times 18!} 5 - \frac{22!}{2! \times 20!} 1 + 1$$

$$\Rightarrow E_{11} = 69348874393137901$$

Calculation, according to the binomial formula for the Euler number, of E_{12}

$$E_{12} = \binom{24}{2} 69348874393137901 - \binom{24}{4} 370371188237525 + \binom{24}{6} 2404879675441 - \binom{24}{8} 19391512145 + \binom{24}{10} 199360981 - \binom{24}{12} 2702765 + \binom{24}{14} 50521 - \binom{24}{16} 1385 + \binom{24}{18} 61 - \binom{24}{20} 5 + \binom{24}{22} 1 - (-1)^{12}$$

$$E_{12} = \frac{24!}{22! \times 2!} 69348874393137901 - \frac{24!}{20! \times 4!} 370371188237525 + \frac{24!}{18! \times 6!} 2404879675441 - \frac{24!}{16! \times 8!} 19391512145 + \frac{24!}{14! \times 10!} 199360981 - \frac{24!}{12! \times 12!} 2702765 + \frac{24!}{10! \times 14!} 50521 - \frac{24!}{8! \times 16!} 1385 + \frac{24!}{6! \times 18!} 61 - \frac{24!}{4! \times 20!} 5 + \frac{24!}{2! \times 22!} 1 - 1$$

$$\Rightarrow E_{12} = 15514534163557086905$$

Calculation, according to the binomial formula for the Euler number, of E_{13}

$$E_{13} = \binom{26}{2} 15514534163557086905 - \binom{26}{4} 69348874393137901 + \binom{26}{6} 370371188237525 - \binom{26}{8} 2404879675441 + \binom{26}{10} 19391512145 - \binom{26}{12} 199360981 + \binom{26}{14} 2702765 - \binom{26}{16} 50521 + \binom{26}{18} 1385 - \binom{26}{20} 61 + \binom{26}{22} 5 - \binom{26}{24} 1 - (-1)^{13}$$

$$E_{13} = \frac{26!}{24! \times 2!} 15514534163557086905 - \frac{26!}{22! \times 4!} 69348874393137901 + \frac{26!}{20! \times 6!} 370371188237525 - \frac{26!}{18! \times 8!} 2404879675441 + \frac{26!}{16! \times 10!} 19391512145 - \frac{26!}{14! \times 12!} 199360981 + \frac{26!}{12! \times 14!} 2702765 - \frac{26!}{10! \times 16!} 50521 + \frac{26!}{8! \times 18!} 1385 - \frac{26!}{6! \times 20!} 61 + \frac{26!}{4! \times 22!} 5 - \frac{26!}{2! \times 24!} 1 + 1$$

$$\Rightarrow E_{13} = 4087072509293123892361$$

Calculation, according to the binomial formula for the Euler number, of E_{14}

$$\begin{aligned}
E_{14} &= \binom{28}{2} 408707250929312892361 - \binom{28}{4} 15514534163557086905 + \\
&\binom{28}{6} 69348874393137901 - \binom{28}{8} 370371188237525 + \binom{28}{10} 2404879675441 - \\
&\binom{28}{12} 19391512145 + \binom{28}{14} 199360981 - \binom{28}{16} 2702765 + \binom{28}{18} 50521 - \binom{28}{20} 1385 + \binom{28}{22} 61 - \\
&\binom{28}{24} 5 + \binom{28}{26} 1 - (-1)^{14} \\
E_{14} &= \frac{28!}{26! \times 2!} 408707250929312892361 - \frac{28!}{24! \times 4!} 15514534163557086905 + \\
&\frac{28!}{22! \times 6!} 69348874393137901 - \frac{28!}{20! \times 8!} 370371188237525 + \frac{28!}{18! \times 10!} 2404879675441 - \\
&\frac{28!}{16! \times 12!} 19391512145 + \frac{28!}{14! \times 14!} 199360981 - \frac{28!}{12! \times 16!} 2702765 + \frac{28!}{10! \times 18!} 50521 - \frac{28!}{8! \times 20!} 1385 + \\
&\frac{28!}{6! \times 22!} 61 - \frac{28!}{4! \times 24!} 5 + \frac{28!}{2! \times 24!} 1 - 1 \\
&\Rightarrow E_{14} = 1252259641403629865468285
\end{aligned}$$

Calculation, according to the binomial formula for the Euler number, of E_{15}

$$\begin{aligned}
E_{15} &= \binom{30}{2} 125225964140362865468285 - \binom{30}{4} 4087072509293123892361 + \\
&\binom{30}{6} 15514534163557086905 - \binom{30}{8} 69348874393137901 + \binom{30}{10} 370371188237525 - \\
&\binom{30}{12} 2404879675441 + \binom{30}{14} 19391512145 - \binom{30}{16} 199360981 + \binom{30}{18} 2702765 - \binom{30}{20} 50521 + \\
&\binom{30}{22} 1385 - \binom{30}{24} 61 + \binom{30}{26} 5 - \binom{30}{28} 1 - (-1)^{15} \\
E_{15} &= \frac{30!}{28! \times 2!} 1252259641403629865468285 - \frac{30!}{26! \times 4!} 4087072509293123892361 \\
&+ \frac{30!}{24! \times 6!} 15514534163557086905 - \frac{30!}{22! \times 8!} 69348874393137901 \\
&+ \frac{30!}{20! \times 10!} 370371188237525 - \frac{30!}{18! \times 12!} 2404879675441 \\
&+ \frac{30!}{16! \times 14!} 19391512145 - \frac{30!}{14! \times 16!} 199360981 + \frac{30!}{12! \times 18!} 2702765 \\
&- \frac{30!}{10! \times 20!} 50521 + \frac{30!}{8! \times 22!} 1385 - \frac{30!}{6! \times 24!} 61 + \frac{30!}{4! \times 26!} 5 - \frac{30!}{2! \times 28!} 1 + 1 \\
&\Rightarrow E_{15} = 441543893249023104553682821
\end{aligned}$$

3. Results

Having carried out the appropriate and proposed calculations, we present the table of the first 20 Bernoulli and Euler numbers listed in Tables 1 and 2, in which the applications of Bernoulli and Euler numbers were carried out separately. The first application corresponds to the Bernoulli numbers, in solving the integral of $\ln(\cos x)$.

Table 1.
The Bernoulli Numbers.

n	B_n
1	1/6
2	1/30
3	1/42
4	1/30
5	5/66
6	691/2730
7	7/6
8	3617/510
9	43867/798
10	174611/330
11	854513/138
12	236364091/2730
13	8553103/6
14	23749461029/870
15	8615841276005/14322
16	7709321041217/510
17	2577687858367/6
18	26315271553053477373/1919190
19	2929993913841559/6
20	261082718496449122051/13530

3.1. For the Bernoulli numbers

$$\int \frac{x}{\sin(ax)} dx = \frac{1}{a^2} \left[ax + \frac{(ax)^3}{3 \times 3!} + \frac{7(ax)^5}{3 \times 5 \times 5!} + \frac{31(ax)^7}{3 \times 7 \times 7!} + \dots + \frac{2(2^{2n-1}-1)}{(2n+1)!} B_n (ax)^{2n+1} + \dots \right] \dots (a)$$

$$\int xt g(ax) dx = \frac{ax^3}{3} + \frac{a^3 x^5}{15} + \frac{2a^5 x^7}{105} + \frac{17a^7 x^9}{2835} + \dots + \frac{2^{2n}(2^{2n}-1)B_n a^{2n-1} x^{2n+1}}{(2n+1)!} + \dots (b)$$

$$\int \frac{tg(ax)}{x} dx = ax + \frac{(ax)^3}{9} + \frac{2(ax)^5}{75} + \frac{17(ax)^7}{2205} + \dots + \frac{2^{2n}(2^{2n}-1)B_n (ax)^{2n-1}}{(2n-1)(2n)!} + \dots (c)$$

$$\int \ln(tgx) dx = x \ln x - x + \frac{x^3}{9} + \frac{7x^5}{450} + \dots + \frac{2^{2n}(2^{2n-1}-1)B_n}{n(2n+1)!} x^{2n+1} + \dots (d)$$

$$\int \ln(\cos x) dx = -\frac{2^{2n-1} \times (2^{2n}-1) \times B_n \times x^{2n+1}}{n(2n+1)!} \dots (e)$$

3.1.1. Application of the Bernoulli Numbers (B_n).

For the expression $\int \ln(\cos x) dx$ (e); we have:

- For: $B_1 = 1/6$

$$\int \ln(\cos x) dx = -\frac{x^3}{6}$$

- For: $B_1 = 1/6$ y $B_2 = 1/30$

$$\int \ln(\cos x) dx = -\frac{x^3}{6} - \frac{x^5}{60}$$

- For: $B_1 = 1/6$, $B_2 = 1/30$ y $B_3 = 1/42$

$$\int \ln(\cos x) dx = -\frac{x^3}{6} - \frac{x^5}{60} - \frac{x^7}{210}$$

- For: $B_1 = 1/6$, $B_2 = 1/30$, $B_3 = 1/42$ y $B_4 = 1/30$

$$\int \ln(\cos x) dx = -\frac{x^3}{6} - \frac{x^5}{60} - \frac{x^7}{210} - \frac{17x^9}{22680}$$

- For: $B_1 = 1/6$, $B_2 = 1/30$, $B_3 = 1/42$, $B_4 = 1/30$ y $B_5 = 5/66$.

$$\int \ln(\cos x) dx = -\frac{x^3}{6} - \frac{x^5}{60} - \frac{x^7}{210} - \frac{17x^9}{22680} - \frac{31x^{11}}{311850}$$

- For: $B_1 = 1/6, B_2 = 1/30, B_3 = 1/42, B_4 = 1/30, B_5 = 5/66$ y $B_6 = 691/2730$.

$$\int \ln(\cos x) dx = -\frac{x^3}{6} - \frac{x^5}{60} - \frac{x^7}{210} - \frac{17x^9}{22680} - \frac{31x^{11}}{311850} - \frac{691x^{13}}{4455}$$

- For: $B_1 = 1/6, B_2 = 1/30, B_3 = 1/42, B_4 = 1/30, B_5 = 5/66, B_6 = 691/2730$ y $B_7 = 7/6$.

$$\int \ln(\cos x) dx = -\frac{x^3}{6} - \frac{x^5}{60} - \frac{x^7}{210} - \frac{17x^9}{22680} - \frac{31x^{11}}{311850} - \frac{691x^{13}}{4455} - \frac{10922x^{15}}{638512875}$$

- For: $B_1 = 1/6, B_2 = 1/30, B_3 = 1/42, B_4 = 1/30, B_5 = 5/66, B_6 = 691/2730, B_7 = 7/6$ y $B_8 = 3617/510$.

$$\int \ln(\cos x) dx = -\frac{x^3}{6} - \frac{x^5}{60} - \frac{x^7}{210} - \frac{17x^9}{22680} - \frac{31x^{11}}{311850} - \frac{691x^{13}}{4455} - \frac{10922x^{15}}{638512875} - \frac{929569x^{17}}{173675502000} + \dots;$$

and so on, the Bernoulli numbers could continue to be applied to other integrals and mathematical expressions in which the Bernoulli numbers are necessary.

The second application corresponds to the Euler numbers, in solving the integral of $[\sec(ax)]/x$.

Table 2.

The Euler numbers.

n	E_n
1	1
2	5
3	61
4	1385
5	50521
6	2702765
7	199360891
8	19391512145
9	2404879675441
10	370371188237525
11	69348874393137901
12	15514534163557086905
13	4087072509293123892361
14	1252259641403629865468285
15	441543893249023104553682821
16	177519391579539289436664789665
17	80723299235887898062168247453281
18	41222060339517702122347079671259045
19	23489580527043108252017828576198947741
20	14851150718114980017877156781405826684425

3.2. For the Euler Numbers

$$\int \frac{x}{\cos(ax)} dx = \frac{1}{a^2} \left[\frac{(ax)^2}{2} + \frac{(ax)^4}{4 \times 2!} + \frac{5(ax)^6}{6 \times 4!} + \frac{61(ax)^8}{8 \times 6!} + \dots + \frac{E_n(ax)^{2n+2}}{(2n+2)(2n)!} + \dots \right] \dots \text{(f)}$$

$$\int x \sec(ax) dx = \frac{1}{a^2} \left[\frac{(ax)^2}{2} + \frac{(ax)^4}{8} + \frac{5(ax)^6}{144} + \frac{61(ax)^8}{5760} + \dots + \frac{E_n(ax)^{2n+2}}{(2n+2)(2n)!} + \dots \right] \dots \text{(g)}$$

$$\int \frac{x}{\cosh(ax)} dx = \frac{1}{a^2} \left[\frac{(ax)^2}{2} - \frac{(ax)^4}{8} + \frac{5(ax)^6}{144} - \frac{61(ax)^8}{5760} + \dots + \frac{(-1)^n E_n(ax)^{2n+2}}{(2n+2)(2n)!} + \dots \right] \dots \text{(h)}$$

$$\int \frac{\sec(ax)}{x} dx = \ln x + \frac{E_n(ax)^{2n}}{2n(2n)!} \dots \text{(i)}$$

3.2.1. Application of the Euler Numbers (E_n).

For example, for the expression $\int \frac{\sec(ax)}{x} dx$ (i); we have:

For $E_1 = 1$

$$\int \frac{\sec(ax)}{x} dx = \ln x + \frac{(ax)^2}{4}$$

For $E_1 = 1$ y $E_2 = 5$

$$\int \frac{\sec(ax)}{x} dx = \ln x + \frac{(ax)^2}{4} + \frac{5(ax)^4}{96}$$

For $E_1 = 1, E_2 = 5$ y $E_3 = 61$

$$\int \frac{\sec(ax)}{x} dx = \ln x + \frac{(ax)^2}{4} + \frac{5(ax)^4}{96} + \frac{61(ax)^6}{4320}$$

For $E_1 = 1, E_2 = 5, E_3 = 61$ y $E_4 = 1385$

$$\int \frac{\sec(ax)}{x} dx = \ln x + \frac{(ax)^2}{4} + \frac{5(ax)^4}{96} + \frac{61(ax)^6}{4320} + \frac{277(ax)^8}{64512}$$

For $E_1 = 1, E_2 = 5, E_3 = 61, E_4 = 1385$ y $E_5 = 50521$

$$\int \frac{\sec(ax)}{x} dx = \ln x + \frac{(ax)^2}{4} + \frac{5(ax)^4}{96} + \frac{61(ax)^6}{4320} + \frac{277(ax)^8}{64512} + \frac{50521(ax)^{10}}{36288000}$$

For $E_1 = 1, E_2 = 5, E_3 = 61, E_4 = 1385, E_5 = 50521$ y $E_6 = 2702765$

$$\int \frac{\sec(ax)}{x} dx = \ln x + \frac{(ax)^2}{4} + \frac{5(ax)^4}{96} + \frac{61(ax)^6}{4320} + \frac{277(ax)^8}{64512} + \frac{50521(ax)^{10}}{36288000} + \frac{540553(ax)^{12}}{1149603840}$$

For $E_1 = 1, E_2 = 5, E_3 = 61, E_4 = 1385, E_5 = 50521, E_6 = 2702765$ y $E_7 = 199360891$

$$\int \frac{\sec(ax)}{x} dx = \ln x + \frac{(ax)^2}{4} + \frac{5(ax)^4}{96} + \frac{61(ax)^6}{4320} + \frac{277(ax)^8}{64512} + \frac{50521(ax)^{10}}{36288000} + \frac{540553(ax)^{12}}{1149603840} + \frac{199360891x^{14}}{1220496076800}$$

For $E_1 = 1, E_2 = 5, E_3 = 61, E_4 = 1385, E_5 = 50521, E_6 = 2702765, E_7 = 199360891$ y $E_8 = 19391512145$

$$\int \frac{\sec(ax)}{x} dx = \ln x + \frac{(ax)^2}{4} + \frac{5(ax)^4}{96} + \frac{61(ax)^6}{4320} + \frac{277(ax)^8}{64512} + \frac{50521(ax)^{10}}{36288000} + \frac{540553(ax)^{12}}{1149603840} + \frac{199360891x^{14}}{1220496076800} + \frac{3878302429(ax)^{16}}{66952927641600} + \dots$$

For $E_1 = 1, E_2 = 5, E_3 = 61, E_4 = 1385, E_5 = 50521, E_6 = 2702765, E_7 = 199360891, E_8 = 19391512145$ y $E_9 = 2404879675441$

$$\int \frac{\sec(ax)}{x} dx = \ln x + \frac{(ax)^2}{4} + \frac{5(ax)^4}{96} + \frac{61(ax)^6}{4320} + \frac{277(ax)^8}{64512} + \frac{50521(ax)^{10}}{36288000} + \frac{540553(ax)^{12}}{1149603840} + \frac{199360891x^{14}}{1220496076800} + \frac{3878302429(ax)^{16}}{66952927641600} + \frac{2404879675441(ax)^{18}}{115242726703104000} + \dots$$

For $E_1 = 1, E_2 = 5, E_3 = 61, E_4 = 1385, E_5 = 50521, E_6 = 2702765, E_7 = 199360891, E_8 = 19391512145, E_9 = 2404879675441$ y $E_{10} = 370371188237525$

$$\int \frac{\sec(ax)}{x} dx = \ln x + \frac{(ax)^2}{4} + \frac{5(ax)^4}{96} + \frac{61(ax)^6}{4320} + \frac{277(ax)^8}{64512} + \frac{50521(ax)^{10}}{36288000} + \frac{540553(ax)^{12}}{1149603840} + \frac{199360891x^{14}}{1220496076800} + \frac{3878302429(ax)^{16}}{66952927641600} + \frac{2404879675441(ax)^{18}}{115242726703104000} + \frac{14814847529501(ax)^{20}}{1946321606541312000} + \dots$$

For $E_1 = 1, E_2 = 5, E_3 = 61, E_4 = 1385, E_5 = 50521, E_6 = 2702765, E_7 = 199360891, E_8 = 19391512145, E_9 = 2404879675441, E_{10} = 370371188237525$ y $E_{11} = 69348874393137901$

$$\int \frac{\sec(ax)}{x} dx = \ln x + \frac{(ax)^2}{4} + \frac{5(ax)^4}{96} + \frac{61(ax)^6}{4320} + \frac{277(ax)^8}{64512} + \frac{50521(ax)^{10}}{36288000} + \frac{540553(ax)^{12}}{1149603840} + \frac{199360891x^{14}}{1220496076800} + \frac{3878302429(ax)^{16}}{66952927641600} + \frac{2404879675441(ax)^{18}}{115242726703104000} + \frac{14814847529501(ax)^{20}}{1946321606541312000} + \frac{69348874393137901(ax)^{22}}{24728016011107368960000} + \dots$$

For $E_1 = 1, E_2 = 5, E_3 = 61, E_4 = 1385, E_5 = 50521, E_6 = 2702765, E_7 = 199360891, E_8 = 19391512145, E_9 = 2404879675441, E_{10} = 370371188237525, E_{11} = 69348874393137901$ y $E_{12} = 15514534163557086905$

$$\int \frac{\sec(ax)}{x} dx = \ln x + \frac{(ax)^2}{4} + \frac{5(ax)^4}{96} + \frac{61(ax)^6}{4320} + \frac{277(ax)^8}{64512} + \frac{50521(ax)^{10}}{36288000} + \frac{540553(ax)^{12}}{1149603840} + \frac{199360891x^{14}}{1220496076800} + \frac{3878302429(ax)^{16}}{66952927641600} + \frac{2404879675441(ax)^{18}}{115242726703104000} + \frac{14814847529501(ax)^{20}}{1946321606541312000} + \frac{69348874393137901(ax)^{22}}{24728016011107368960000} + \frac{3102906832711417381(ax)^{24}}{2978152328319549308928000} + \dots$$

and so on, the Euler numbers could continue to be applied.

4. Discussion

Bernoulli numbers decrease until the third position and then increase indefinitely, coinciding with what Fernández [5] and Vera [6]; report; they are the smallest possible ratio between two relatively prime numbers, coinciding with what Benton mentions, [1] and that Bernoulli numbers have numerous applications in the field of applied mathematics, mathematical analysis, and engineering, such as derivatives and integrals, as mentioned by Ruíz [10].

When calculating Bernoulli numbers, these can be applied in a wide range of mathematical applications, initially to the concepts of definite and indefinite integrals of trigonometric, inverse, and logarithmic functions, and consequently in the mathematical modeling of a wide range of real cases involving transcendental function relationships; coinciding with what was mentioned by Bronshtein and Semendiaev [7].

The estimated Euler numbers can be used in the integral calculation of functions and expressions more complicated than those involving Bernoulli numbers. Therefore, their application and solution to different cases are more rigorous, enabling the solving of problems of greater difficulty and providing results that are closer to reality; coinciding with what was mentioned by Murray [22].

Euler numbers are all positive, agreeing with what Varadarajan [20] and Simmons [23] mention, at least from position one to position twenty. Like Bernoulli numbers, Euler numbers also participate in a large number of mathematical relationships and physical phenomena applied to technological, scientific, and engineering processes, such as those mentioned by Abramowitz and Stegun [4]. It is concluded that all the Euler numbers calculated are integers and positive, that the sum of two consecutive Euler numbers is divisible by 3 (three), that Euler numbers alternately end in 1 or 5, that Bernoulli numbers are all positive, that Bernoulli numbers, from the fourth position onward, grow indefinitely, that Bernoulli numbers are the smallest ratio between two relatively prime numbers, and there are other more important direct applications to the mathematics of Bernoulli and Euler numbers that appear in a wide variety of processes and that are truly the essence of engineering, applied mathematics, experimental models, and their various behaviors. Likewise, it is recommended, where possible, to design software and/or special computer programs to obtain Bernoulli and Euler numbers from twenty onward, in the manner indicated by the authors.

Transparency:

The authors confirm that the manuscript is an honest, accurate, and transparent account of the study; that no vital features of the study have been omitted; and that any discrepancies from the study as planned have been explained. This study followed all ethical practices during writing.

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