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On some covering properties via a- open sets

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Abstract: In this paper the notion of a-open set used as a tool to introduce certain types of covering properties which is similar to the familiar property of Hurewicz, and prove that we can use a-open sets instead of open sets in the definition of a-compact and a-Hurewicz space and investigated. Some properties and counter examples are given, also the relationship between these spaces was considered.
Keywords: *A-compact space, A-Hurewicz space, A-open set, Covering property.*

1. Introduction

The classical Hurewicz property has a long history from the paper [1]. A topological space X has Hurewicz property if for each sequence $(U_n)_{n\in\mathbb{N}}$ of open covers of X there exists a sequence $(V_n)_{n\in\mathbb{N}}$ where for each $n \in \mathbb{N}$ k, V_n is a finite subset of U_n such that for each $x \in X$, $x \in \bigcup V_n$ for all but finitely many n.

Recently, several weak variants of Hurewicz property have been studied after applying the interior and the closure operators in the definition of a Hurewicz Property. Also, the other ways have been examined when the sequence of open covers are replaced with generalized open sets. For the study of the variants of Hurewicz spaces, the readers can see [2, 3, 4, 5].

Some types of sets play an important role in the study of various properties in topological spaces. Many authors introduced and studied various generalized properties and conditions containing some forms of sets in topological spaces. In this paper, we investigate some properties of a-open sets. Moreover, the relationships among open sets, a-open sets and the related classes of sets are investigated.

In this paper, spaces X and Y mean topological spaces. For a subset A of a space X, $cl(A)$ and $int(A)$ represent the closure of A and the interior of A, respectively.

In this paper, we examine the covering properties namely: a-Hurewicz, which is a like to the classical Hurewicz property by using a-open sets see [6, 7, 8, 9]. The following generalizations of open sets will be used for definitions of variations on the Hurewicz property:

The paper is organized in such a way that after this introduction in section two we give information about terminology and notation. In section 3and 4 we show that we can replace open sets with a-open sets in the definition of a-Hurewicz spaces. Also, we investigate the behavior of a-Hurewicz properties with respect to subspaces, products and a-continuous image.

2. Background Material

The interior and closure operators in topological spaces play a vital role in the generalization of open sets and closed sets. The relations on the interior and closure operators motivate the point set topologists to introduce several forms of a-open sets and a-closed sets. Some of them are given in the next definition.

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Definition 2.1: [6,7, 8] A subset U of a $T.s\ (X, \mathcal{T})$ is called:
i. regular open (r-open), if $A = int(cl(A))$.
ii. δ-interior of a subset A of X is the union of all r-open set of X contained in A and it is denoted by δ-$int(A)$.
iii. δ-open if $A = \delta\text{-}int(A)$.
iv. The δ-closure of a set A in X denoted $\delta\text{-}cl(A)$ and defined by:
$\{ x \in X : A \cap int(cl(B)) \neq \emptyset, B \in \mathcal{T} \text{ and } x \in B\}$.
iv. a-open, if $A \subseteq int(cl(\delta\text{-}int(A)))$.

Example 2.2:
i. In $(\mathbb{R}, \mathcal{T}_U)$ a subset $(0, 1)$ is an a-open.
ii. Let $X = \{a, b, c\}$ with $\mathcal{T} = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$. A subset $\{b\}$ is not a-open.

Theorem 2.3: [6] A subset U of a $T.s\ (X, \mathcal{T})$ is an a-open, if and only if for each $x \in U$ there exists δ-open set P of X such that $x \in P \subseteq U$.

Remark 2.4: [6]
i. The family of all a-open sets of a $T.s\ (X, \mathcal{T})$ forms a topology on X, denoted by $\mathcal{T}^a$.
ii. For any subset of a $T.s\ (X, \mathcal{T})$, we conclude the following diagram:

$$r\text{-open} \Longrightarrow \delta\text{-open} \Longrightarrow a\text{-open} \Longrightarrow \text{open}$$

Definition 2.5: For any subset A a $T.s\ (X, \mathcal{T})$, the following symbols denote:
i. $cl_a(A)$ is the intersection of all a-closed subsets of X containing A.
ii. $int_a(A)$ is the union of all a-open subsets of X contained in A.
iii. A is said to be a-dense, if $cl_a(A) = X$.
Recall that A mapping $f: (X, \mathcal{T}) \to (Y, \mathcal{T}')$ is said to be δ-continuous, if $f^{-1}(U)$ is δ-open set of X for every open set U of Y, [9].

Definition 1.6: Let $(X, \mathcal{T})$ and $(Y, \mathcal{T}')$ be two $T.s$'s. Then a mapping $f: (X, \mathcal{T}) \to (Y, \mathcal{T}')$ is said to be:
i. a-continuous, if $f^{-1}(V)$ is a-open set of X for every open set V of Y.
ii. a-irresolute continuous, if $f^{-1}(V)$ is a-open set of X for every a-open set V of Y.

Example .2.7: Let $f : (\mathbb{N}, \mathcal{T}_{ind}) \longrightarrow (\mathbb{N}, \mathcal{T}_{cof})$ be a mapping which is defined by $f(x) = x$ for all $x \in \mathbb{N}$. Then f is a-continuous.

3. a-Compact Space

In this section, we introduce the concept of a-compact spaces along with some basic properties of it. Begin this section by giving some properties of a-irresolute continuous and a-continuous mappings. The prove of the following propositions is obvious and so omitted

Definition 3.1: Let $f: (X, \mathcal{T}) \to (Y, \mathcal{T}')$ be a mapping. Then f is said to be a-open (a-closed, resp.) mapping, if $f(U)$ is a-open (a-closed) set of Y for every open (closed , resp.) set U of X.

Proposition 3.2: Let $f: (X, \mathcal{T}) \to (Y, \mathcal{T}')$ be a mapping. Then the following statements are equivalent:
i. f is a-irresolute continuous.
ii. $f^{-1}(F)$ is a-closed set of X for every a-closed set F of Y.
iii. $cl_a(f^{-1}(B)) \subseteq f^{-1}(cl_a(B))$ for all $B \subseteq Y$.
iv. $f(cl_a(A)) \subseteq cl_a(f(A))$ for all $A \subseteq X$.

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v. $f^{-1}\left(int_{a}(B)\right) \subseteq int_{a}\left(f^{-1}(B)\right)$ for every $B \subseteq Y$.

Proposition 3.3:
i. Every continuous mapping is a-continuous .
ii. Composition of two a-irresolute continuous mappings is a-irresolute continuous.
iii. Composition of a-irresolute continuous and a-continuous mappings is a-continuous.

Definition 3.4: A space X is said to be a-compact(resp. a- Lindelof), if every a-open cover of X by a-open subset of X has a finite (resp. countable) subcover.

Proposition 3.5 : The intersection of δ - open and an a- open sets is an a-open .

Proof: Let A is δ-open and B is an a - open sets in $\mathcal{T}_x$. To show that A ⋂ B is an a-open in $\mathcal{T}_y$.

A ⋂ B $\subseteq$ δ- int (A) ⋂ int (cl (δ- int (B))).
$\subseteq int_A$(δ- int(A) ⋂ int (cl (δ- int (B))) .
$\subseteq int_A$(cl (δ- int(A) ⋂ int (cl (δ- int (B))).
$\subseteq int_A$(cl(δ- int (A ⋂ B))) .
$\subseteq int_A$(cl(δ- int_A(A ⋂ B))) .

Since int_A(cl(δ- int_A(A ⋂ B))) is a-open set in $\mathcal{T}_y$,so int_A(cl(δ- int_A(A ⋂ B))) = int_A(cl(δ- int_A(A ⋂ B) ⋂ B))). Thus A ⋂ B $\subseteq int_A$(cl_A (δ- int_A(A ⋂ B))) .

Theorem 3.6: A δ -open subset Y of a space X is a-compact if and only if every a-open cover of Y by the a-open subset of X has a finite subcover.

Proof: Let Y be a-compact subset of X. Let $\{G_\lambda : \lambda \in \Lambda\}$ be a-open cover of Y, where each G_λ is a-open set in X for all $\lambda \in \Lambda$. Then, $Y \subseteq \bigcup_{\lambda\in\Lambda} G_\lambda$ that is $Y \subseteq \bigcup_{\lambda\in\Lambda} G_\gamma \cap Y$, where each $G_\lambda \cap Y$ is a-open in T_Y by the Theorem (3.5). Therefore, by a-compactness of Y, there is a finite subcollection Λ_0 of Λ with $Y \subseteq \bigcup_{\lambda\in\Lambda_0} G_\lambda \cap Y$, so $Y \subseteq \bigcup_{\lambda\in\Lambda_0} G_\lambda$. Thus, if Y is a-compact then every a-open cover of Y by the a-open set of X has a finite subcover. Conversely, let $\{Y_\lambda : \lambda \in \Lambda\}$ an a-open cover of Y by the a-open sets of Y.Thus, $Y \subseteq \bigcup_{\lambda\in\Lambda} Y_\lambda$. Since Y is open, Y_λ is a-open set in X for all $\lambda \in \Lambda$. So, $\{Y_\lambda : \lambda \in \Lambda\}$ is a-open cover of Y by the a-open sets of X. Then by the given condition, there is a finite subcover Λ_0 of Λ such that $Y \subseteq \bigcup_{\lambda\in\Lambda} G_\lambda$. So by the definition of a-compact space, Y is a-compact. Hence, this completes the proof.

Theorem .3.7: An a-closed subset of an a-compact space is a-compact.

Proof: Let $(X ,\mathcal{T})$ be a a-compact topological space and let Y be an a-closed subset of X. Now we have to show that, Y is a-compact. Let $\{G_\lambda : \lambda \in \Lambda\}$ be an a-open cover of Y , where each G_λ is a-open set in $(X ,\mathcal{T})$ for all $\lambda \in \Lambda$. Then $Y \subseteq \bigcup_{\lambda\in\Lambda} G_\lambda$. So, $X \subseteq (X \setminus Y) \cup (\bigcup_{\lambda\in\Lambda} G_\lambda)$. Since X is a-compact, there exists a finite collection Λ_0 of Λ such that $X \subseteq (X \setminus Y) \cup (\bigcup_{\lambda\in\Lambda_0} G_\lambda)$ and so $Y \subseteq \bigcup_{\lambda\in\Lambda_0} G_\lambda$. Hence every a-open cover $\{G_\lambda : \lambda \in \Lambda\}$ of Y has a finite subcover. Then Y is an a-compact. Hence, the theorem is done.

Theorem 3.8: Let f be a-continuous mapping from $(X,\mathcal{T})$ to $(Y,\mathcal{T}')$ and V be a a-open set in Y. Then $f^{-1}(V)$ is an a-open in X.

Proof: Let $f: (X,\mathcal{T}) \to (Y,\mathcal{T}')$ be a-continuous mapping. We show that, $f^{-1}(V)$ is a-open in X.

Since V be a δ-open set in Y, then from δ-continuity of f, we must have $f^{-1}(V)$ is an open set of X. That is, for every $x \in f^{-1}(V)$, there exists a a-open set W in X with $x \in W \subseteq f^{-1}(V)$.Let $x \in f^{-1}(V)$. Then $f(x) \in V$, since V is a-open, there exists a a-open set U in Y such that $f(x) \in U \subseteq V$. So, $x \in f^{-1}(U) \subseteq f^{-1}(V)$. Now since U is a-open in Y and f is a-open, a-continuous mapping, Then, $f^{-1}(U)$ is a-open set in X. Hence $f^{-1}(V)$ is a-open in X, therefore $f^{-1}(V)$ is an a-open in X.

Theorem 3.9: Let $f: (X,\mathcal{T}) \to (Y,\mathcal{T}')$ be an a-open, a-continuous mapping, X is an a-compact. Then $f(X)$ is an a-compact subset Y.

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Proof: Let $\{V_\lambda : \lambda \in \Lambda\}$ be an a-open cover of $f(X)$ in Y, so $f(X) \subseteq \bigcup_{\lambda\in\Lambda} V_\lambda$ and hence equality is hold, $X \subseteq f^{-1}(\bigcup_{\lambda\in\Lambda} V_\lambda) = \bigcup_{\lambda\in\Lambda} V_\lambda\ f^{-1}(V_\lambda)$. Since f is a-open and a-continuous by the previous Theorem (3.8), each $f^{-1}(V_\lambda)$ is a-open in X. Thus $\{f^{-1}(V_\lambda) : \lambda \in \Lambda\}$ is a a-open cover of X. Consequently, and from a-compactness of X, there exists $\{f^{-1}(V_{\lambda_i})\ i = 1, 2, \ldots, m\}$ of $\{f^{-1}(V_\lambda) : \lambda \in \Lambda$ which also covers X. Thus $X \subseteq \bigcup_{i=1}^{m} f^{-1}(V_{\lambda_i})$. So, $f(X) \subseteq \bigcup_{i=1}^{m} V_{\lambda_i}$. Therefore, $\{V_{\lambda_i} : i = 1, 2, \ldots, m\}$ is a finite subcollection of $\{V_\lambda : \lambda \in \Lambda\}$ which covers $f(X)$. so, $f(X)$ is a-compact in $(Y, \mathcal{T}')$.

Definition 3.10: Let $(X, \mathcal{T})$ be a T.s and $\mathcal{A}$ be a family of subsets of $\mathcal{T}^a$. Then a-star of $D \subseteq X$ with respect to $\mathcal{A}$ is the set:

$St_a(\mathcal{A} ، D) = \bigcup\{Q \in \mathcal{A} : Q \cap D \neq \emptyset\}$.

Remark 3.11: A a-star of a singleton set $\{x\}$, $x \in X$ with respect to $\mathcal{A}$ is said to be a a-star of a point and defined as:

$St_a(\mathcal{A}, \{x\}) = \bigcup\{Q \in \mathcal{A} : Q \cap \{x\} \neq \emptyset\}$.

Example 3.12: For any non-empty set X. Then:

i. In $(X, \mathcal{T}_{dis})$, it follows that $\mathcal{T}^a = \mathcal{T}_{dis}$ and $St_a(\mathcal{A} ، \{x\}) = X$ for any $x \in X$.

ii. In $(X, \mathcal{T}_{ind})$, it follows that $\mathcal{T}^a = \mathcal{T}_{dis}$ and $St_a(\mathcal{A} ، X) = X$.

Theorem 3.13: Let $(X, \mathcal{T})$ be a T.s and U be an a-open of X. For every a-dense subspace $Y \subseteq X$ there exists a subset $D \subseteq Y$ such that $St_a(D, U) = X$.

Proof: From a-density of Y, we have $St_a(D, U) = \bigcup\{O \subseteq Y : D \cap O \neq \emptyset\}$. That is, for any U an a-open of X, we have $St_a(D, U) = X$.

Remark 3.14: For every T.s $(X, \mathcal{T})$ and D is a a-open cover of X. It follows that $St_a(D, U)$ is an a-open set of X.

Definition 3.15: A T.s $(X, \mathcal{T})$ is said to be :

- a-star- compact , if for every a-open covering $\mathcal{U}$ of X, there exists a finite subset F of $\mathcal{U}$ such that $St_a(\bigcup F, \mathcal{U}) = X$.
- strong a-star- compact , if for every a-open covering $\mathcal{U}$ of X , there exists a finite subset F of X such that $St_a(F, \mathcal{U}) = X$.
- a - star- Lindelof , if for a –open cover $\mathcal{U}$ of X ,there exists countable subset F of $\mathcal{U}$ such that $St_a(\bigcup F, \mathcal{U}) = X$.
- strong a - star- Lindelof , if for a –open cover $\mathcal{U}$ of X ,there exists countable subset F of $\mathcal{U}$ Such that $St_a(F, \mathcal{U}) = X$.

Theorem 3.16: Let $(X, \mathcal{T})$ be a a-compact space and O any a-open covering of X. Then there exists a finite subset F of X such that $St_a(F, \mathcal{U}) = X$ and hence $(X, \mathcal{T})$ is a-star compact.

Proof: Suppose that for each finite set F of X such that $St_a(F, \mathcal{U})$ is a proper subset of X, such that $F = \{x_1, x_2,, \ldots, x_n\}$. Suppose that a set $B = \{x_1, x_2,, \ldots, x_n, \ldots\} \subseteq X$ and for each $n \geq 1$, it follows that $x_{n+1} \notin St_a(F, \mathcal{U})$. Let $y \in cl_a(B)$. Then $B \cap U \neq \emptyset$ for some $U \in \mathcal{U}$, where $y \in U$. Let η be with $x_n \in U$ such that $y \in St_a(\{x_1, x_2,, \ldots, x_\eta\}, \mathcal{U})$. Then $\{St_a(\{x_1, x_2,, \ldots, x_n\}, \mathcal{U}) : n \geq 1\}$ is a a-open cover of $cl_a(B)$. Consequently, $cl_a(B)$ is a a-compact set. But, by construction of a set B, and so $\{St_a(\{x_1, x_2,, \ldots, x_n\}, \mathcal{U}) : n \geq 1\}$ has no finite a-cover. This contradiction establishes the theorem.

Theorem 3.17: Every a-compact topological space is strong a-star- compact space .

Proof: Let $\mathcal{W}$ be a-open cover of a- compact space X. Then there exists a finite subset $\mathcal{W}' = \{W_1, W_2,, \ldots, W_n, \ldots\} \subseteq \mathcal{W}$ such that $\bigcup \mathcal{W}' = \bigcup_{i=1}^{k} W_i = X$. Now take $x_i \in W_i$ for each i=1,2,…,k and from a finite set $F = \{x_1, x_2,, \ldots, x_k\}$, then $X = St_a(F, \mathcal{W}) \subseteq St_a(F, \mathcal{W}') = X$,

There for X is a strong a-star- compact space .

Example 3.18: Converse of the above theorem may not be true .

Consider X the set of natural numbers and the topology $\tau=\{1,2,3,\ldots,n\}:\in n\in\mathbb{N}\}\cup\{x,\emptyset\}$ on X ,Then for a finite subset F = {1} $\subseteq X$ and $\mathcal{W}$ be an arbitrary a- open cover of X,We have $St_a(F,\mathcal{W})=\bigcup\mathcal{W}=X$ .Hence X is strong a-star- compact space. On the other hand , let
$\mathcal{W}=\{\mathcal{W}_n=\{1,2,3,\ldots,n\}:n\in\mathbb{N}\}$ is an $a-$open cover of X.
Suppose $\mathcal{W}'$ is a finite subcover of it . By the construction of X, we can find a largest set $\mathcal{W}_\lambda\in\mathcal{W}'$,where $\lambda\in\mathbb{N}$,so $\bigcup\mathcal{W}=\mathcal{W}_\lambda=\{1,2,3,\ldots,\lambda\}$.
Then$\{\lambda+1,\lambda+2,\lambda+3,\ldots\}$remains without cover .Thus, there is no finite subcover for $\mathcal{W}$,so X is not a- compact space.

4. a-Hurewicz Spaces

Definition 4.1: Let $(X,\mathcal{T})$ be a T.s. and $A\subseteq X$. Then A is said to have the a-Hurewicz property, if for any sequence $(U_n)_{n\in\mathbb{N}}$ of a-open covers of A, there is a sequence $(V_n)_{n\in\mathbb{N}}$ for any $n\in\mathbb{N}$, V_n is a finite subset of U_n and for each $x\in A$ for all but finitely many n, with $x\in\bigcup V_n$. We say that X is a- Hurewicz space, if the set X is a-Hurewicz.

Example 4.2: Every a-compact space is a- Hurewicz space. The convers is not true . Let X = R with the topology $\mathcal{T}=\{U\subseteq X: U=\emptyset$ or $X\setminus U$ is countable $\}$ is a T_1 a-Hurewicz space which is not a- compact.

In the following theorem ,we put a condition to show that a subspace of the a- Hurewicz space is also satisfied .

Theorem 4.3: Let X be the a-Hurewicz space and Y is a-clopen subspace of X ,then Y is the a-Hurewicz space .

Proof : suppose that Y is a-clopen subspace of the a- Hurewicz space and let $(\mathcal{U}_n)_{n\in\mathbb{N}}$ be a sequence of a-open covers of Y. It easy to see that every a-open subset of $a-clopen\,Y$ is the intersection of a-open subset of X with Y. Then, for each n $\in N$ and each $\mathcal{U}\in\mathcal{U}_n$ there exists an a-open set $\mathcal{G}_u$ in X such that $\mathcal{U}=Y\cap\mathcal{G}_u$. Let $\mathfrak{E}_n=\{\mathcal{G}_u: \mathcal{U}\in\mathcal{U}_n\}\cup\{X\setminus Y\}$, $n\in N$. Then$(\mathfrak{E}_n)_{n\in N}$ is a sequence of a-open covers of X . The a- Hurewiczness property of X , implies The existence of a sequence $(\mathcal{W}_n)_{n\in N}$ with $\mathcal{W}_n$ is a finite subset of $\mathfrak{E}_n$ for each $n\in N$ and $X=\bigcup_{n\in N}\bigcup\mathcal{W}_n$. If we put for each n ,$\mathcal{V}_n=\{\mathcal{U}: \mathcal{G}_u\in\mathcal{W}_n\}$, we obtain the sequence $(\mathcal{V}_n)_{n\in N}$ is a finite subset of $\mathcal{U}_n$and each $x\in Y$ for all but finitely many n , with $x\in\bigcup\mathcal{V}_n$,That is Y an a-Hurewicz space.

The a-Hurewiczness is an a-topological property, as evidenced by the following theorem.

Theorem 4.4 : An a- irresolute image of an a-Hurewicz space is a Hurewicz space .

Proof : Let X be an a-Hurewicz space and $Y=f(X)$ its image under a- continuous mapping $f: X\longrightarrow Y$. Let $(\mathcal{V}_n)_{n\in N}$ be a sequence of a- open covers of Y and $x\in X$. Since f is a- irresolute,

Setting $\mathcal{U}_n=f^{-1}(\mathcal{V}_n)$, $n\in N$,we get the sequence $(\mathcal{U}_n)_{n\in N}$ of a- open covers of X .Use the fact X is a- open covers and for each n , find a finite subset $\mathcal{H}_n$ of $\mathcal{U}_n$ with for each $x\in X$. For all but finitely many $n\in N$, such that $X=\bigcup_{n\in N}\bigcup\mathcal{H}_n$.Let $\mathcal{W}_n=f(\mathcal{H}_n), n\in N$. Then the sequence $(\mathcal{W}_n)_{n\in N}$ verifies for $(\mathcal{V}_n)_{n\in N}$ that Y is a-Hurewicz space.

A-topological property is a property maintained by a-homeomorphisms.

Theorem 4.5 : If X is an a- Hurewicz space and Y a- compact space , then $X\times Y$ is a-Hurewicz space .

Proof : Let $X=\bigcup\{X_k: k\in N\}$,where each X_k is a- Hurewicz . Let $\{\mathcal{U}_n: n\in N\}$ be a sequence of a -open covers of X. For each $k\in N$, take the sequence $\{\mathcal{U}_n: n\geq k\}$ .For each $k\in N$, since X_k is a-Hurewicz ,there are a dense subset S_k of X_k and a sequence$(\mathcal{V}_{n,k}: n\geq k)$ such that for each $n\geq k$, $(\mathcal{V}_{n,k}$ is a finite subset of $\mathcal{U}_n$and for each $x\in S_k$, $x\in\bigcup\mathcal{V}_{n,k}$ for all but finitely many $n\geq k$. Let S =

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$\bigcup_{k\in N} S_k$.Then S is a dense subset of X. For each $n \in N$, let $\bigcup\{(\mathcal{V}_{n,j} : j \leq n\}$.Then each $\mathcal{V}_n$ is finite subset of $\mathcal{U}_n$.The dense subset S of X and the sequence $(\mathcal{V}_n : n \in N)$ witness that X is an a- Hurewicz space ;If for each $x \in$ S, there exists some $k \in N$ such that $x \in S_k$,then $x \in \bigcup\mathcal{V}_n$ for all but finitely many $n \geq k$.

Remark 4.6 : The product of a- Hurewicz space and a- compact is a- Hurewicz space, as demonstrated by the previous theorem.

Proposition 4.7: a-int (A) ⊆ int (A).

Proof: Let x ∈ a-int (A) , there is a - open set B such that x ∈ B ⊆ A. So, there is open set B such that x ∈ B ⊆ A . Then x ⊆ int (A) .Hence, (A) ⊆ int (A).

Proposition 4.8: Every a-open set is open.

Proof: Let A is a-open set . Suppose x ∈ A . Then, there exists $\delta -$ open set B such that x ∈ B ⊆ A. Since B is open. Then, for every x ∈ A there exists an open set B such that x ∈ B ⊆ A. Hence A is open set.

Proposition 4.9: If A = a-int (A) then A is an a-open set.

Proof: a-int (A) = { ∪ B:B ⊆ A, B is an a-open set}. Let A = a-int (A) to show A is a-open set. It is clearly that a-int (A) ⊆ A. Conversely, suppose A ⊆ int (cl(δ-int(A))) such that B= cl(δ-int(A)). Then A ⊆ int (B). Then, a-int (A) ⊆ int (B). So, A ⊆ int (cl(δ-int(A))) . Hence A is $a -$ open set.

A mapping $f : X \longrightarrow Y$ is called contra a- continuous if the preimage of each a-open set in Y is a-closed in X .A mapping f is called pre- a- continuous if $f^{-1}(U) \subset a$- Int (a-cl($f^{-1}(U)$) whenever U is a-open in Y.

Theorem .4.10 : A contra - a- continuous , pre- a- continuous image of Y of an a- Hurewicz space X is an a- Hurewicz space.

Proof: Let (U_n: $n \in N$) be a sequence of a-open covers of Y. Since f is contra - a- continuous ,for each $n \in N$ and for each $U \in U_n$ the set $f^{-1}(U)$ is a-closed in X . Since f is pre- a- continuous $f^{-1}(U) \subset$ a- Int (a-cl($f^{-1}(U)$),so that $f^{-1}(U) \subset$ a- Int($f^{-1}(U)$) .On the other hand, a- Int($f^{-1}(U)$) $\subset f^{-1}(U)$,hence$f^{-1}(U)$= a - Int (a-cl($f^{-1}(U)$). Therefore, for each n , the set V_n={$f^{-1}(U)$: $U \in U_n$} is a cover of X by a-open sets. Since X is a- Hurewicz space there is a sequence(g_n: $n \in N$) such that for each n, g_n is finite subset of V_n and each $x \in X$ belongs to ⋃{ a-cl(G): G ∈ g_n}.Hence $\mathcal{W}_n$= { f(G): G ∈ g_n } is a finite subset of U_n for each $n \in N$ and each $z \in Y$ belongs to a-cl($\bigcup\mathcal{W}_n$) for all but finitely many n . This just means that Y is an a- Hurewicz space .

Definition 4.11: A topological space X is:

- star a- Hurewicz space if it satisfies: For each sequence of elements of a-open cover (U_n: $n \in N$) there is sequence (V_n: $n \in N$) such that for each $n \in N$, V_n is finite subset of U_n ,and each $x \in X$ belong to $St_a(\bigcup V_n , U_n)$ for all but finitely many in.
- strong star a- Hurewicz space if it satisfies: For each sequence of elements of a-open cover (U_n: $n \in N$) there is sequence (A_n: $n \in N$) such that for each $n \in N$, A_n is finite subset of X ,and each $x \in X$ belong to $St_a(A_n , U_n)$ for all but finitely many in.

Now, we can form the following diagram.

Strong star a- compact ⟹ Strong star a-Hurewicz ⟹ Strong star a- Lindelof
⇓ ⇓ ⇓
star a- compact ⟹ star a-Hurewicz ⟹ star a- Lindelof

A space X is said to be σ-strongly star a-compact if it can be expressed as the union of countably many σ-strongly a-compact spaces.

Theorem 4.12: Every σ-strongly star a- compact space is strong star a- Hurewicz space.

Poof : Let σ-strongly star a- compact space .suppose that $Y = \bigcup_{n\in N} Y_n$, where each Y_n is strongly Star a- compact . Let $Y_1 \supset Y_2 \supset \ldots \supset Y_n \supset \ldots$, since the union of finitely many strongly star a-

Compact spaces remains strongly star α- compact . Let $(Y_n: n \in N)$ be a sequence of α-open cover of Y .For each $n \in N$ let A_n be a finite subset of Y_n such that $St_a(A_n , Y_n) \supset Y_n$.It follows that each point of Y belongs to all but finitely many sets of $St_a(A_n , Y_n)$.By the sequence $(A_n: n \in N)$ we have Y is strong star α- Hurewicz space.

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