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Factors affecting mobility of harmonic symbols

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Abstract: In this paper, we have illustrated the Bergmann domains and the Toeplitz operators with their symbiotic symbols in some special domains in order to clarify the properties that correspond to the fixed values. And we characterize the bounded harmonic functions for which the Toeplitz operators Bergman space are essentially commuting.

Keywords: *Bergman, Correspond, Function, Functions, Harmonic bounded, Invariant, Operator, Spaces, Toeplitz, Value.*

1. Introduction

Suppose that dA stands for the measure of the space defined on the open unit disk in D of the complex plane $L^2(D, dA)$ represents the inner Hilbert space

$$\langle f, g\rangle = \int_D f\bar{g}dA$$

Bergmann space L_a^2 is a set of function $L^2(D, dA)$ which properties that are included in it analytically$D$. Which means that Bergmann's space L_a^2 it is a closed subspace$L^2(D, dA)$, and so there is an orthogonal projection P from $L^2(D, dA)$ toL_a^2, [1,6]. For$(\varphi + 1) \in L^\infty(D, dA)$, the Toeplitz effects with symbol$(\varphi + 1)$, denoted $T_{\varphi+1}$ is operator from L_a^2 to L_a^2 knowledge before$T_{\varphi+1}f = P\{(\varphi + 1)f\}$. By harmonic function, we mean a function with a complex value over D for which the Laplacian congruent is zero.

Theorem 1. Suppose that $(\varphi + 1)$ and $(\psi + 1)$ definite harmonic functions on$D$. So that

$$T_{\varphi+1}T_{\psi+1} = T_{\psi+1}T_{\varphi+1}$$

If and only if

a- $(\varphi + 1)$ And $(\psi + 1)$ are both analytic on D.
b- $(\bar{\varphi})$ and $(\bar{\psi})$ also analytic on D.
c- There are constants $a, b \in C$ not both equal to, such that $a(\varphi + 1) + b(\psi + 1)$ is constant in$D$.

We will see if the direction of that theorem is trivial, but proving the direction "only if" requires an inverse of the invariant form of the mean-value property.

The clarification of Theorem 1 similar to a similar result installed in [2,7] for Toeplitz effects in the symbol $L^\infty(\partial D)$ acts on Hardy space$H^2(\partial D)$. Brown and Holmes prove these results through the effect matrix of Hardy spaces in Bergman spaces. L_a^2, Toeplitz effects do not have good matrices, and the techniques used by Brown and Holmes do not seem to work in this context. Thus function theory, rather than matrix manipulation, plays a large role in our proof.

A special case of Theorem 1 was proved in [3.8], using function theory techniques quite different from those that we use here. Also proved a special case of Theorem 1; our proof makes use of some of his ideas.

Functions in $L^\infty(\partial D)$ correspond, via the Poisson integral, to bounded harmonic functions on$D$, so the restriction in Theorem 1 to consideration only of Toeplitz effects with harmonic symbols is natural.

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More importantly, Theorem 1 does not hold if "We can replace measurable harmonic ". For example, Paul Bourdon has pointed out to us that if $(\varphi+1)$ and $(\psi+1)$ are any two radial functions in$L^{\infty}(D, dA)$, then

$T_{\varphi+1}T_{\psi+1}=T_{\psi+1}T_{\varphi+1}$(A function is called radial if its value at z depends only on$|z|$). Thus, the following open problem may be hard: Find conditions on functions $(\varphi+1)$ and $(\psi+1)$ in $L^{\infty}(D, dA)$that are necessary and sufficient for $T_{\varphi+1}$to commute with $T_{\psi+1}$

2. The Constant Property of the Argument Value

A continuous function on the disk D is harmonic if and only if it has the mean value property. We characterize harmonicfunctions in terms of an invariant mean value property.

Let $Aut(D)$ denote the set of analytic, one-to-one maps of D onto D (Where Aut stands for Automorphic). A function h on D is in $Aut(D)$ if and only if there exist $\alpha \in \partial D$ and $\beta \in D$ such that

$$h(z)=\alpha\frac{\beta-z}{1-\bar{\beta}}$$

for all $z \in D$

A function $u \in C(D)$ is said to have the invariant mean value property if

$$\int_0^{2\pi} u\left(h\left(re^{i\theta}\right)\right)\frac{d\theta}{2\pi}=u\big(h(0)\big) \qquad (1)$$

for every $h \in Aut(D)$ and every $r \in [0,1)$. Here "invariant" refers to conformal invariance, meaning invariance under composition with elements of$Aut(D)$. If u is harmonic on D, then so is $u^{\circ}h$ for every$h \in Aut(D)$; thus harmonic functions havethe invariant mean value property. The converse is also true [4,6], if a function $u \in C(D)$ has the invariant mean value property, then u is harmonic on$D$.

The invariant mean value property concerns averages over circles with respect to arc length measure. Because we are dealing with the Bergman space, L_a^2 we need an invariant condition stated in terms of an area average over$D$. Thus we say that a function $u \in C(D) \cap L^1(D, dA)$ has the area version of the invariant mean value property if

$$\int_D uoh\frac{d\theta}{2\pi}=u\big(h(0)\big) \qquad (2)$$

for every $h \in Aut(D)$. If u is in$C(D) \cap L^1(D, dA)$, then so is $u^{\circ}h$ for every$h \in Aut(D)$, so the left-hand side of the above equation makes sense. Note that the area version of the invariant mean value property deals with integrals over all of$D$, asopposed to integrals over rD for$r \in (0,1)$.

If u is harmonic on D and in$L^1(D, dA)$, then so is $u^{\circ}h$ for every$h \in Aut(D)$. Thus, by the mean value property, harmonic functions have the area version of the invariant mean value property. Whether or not the converse is true is an open question. In other words, if $u \in C(D) \cap L^1(D, dA)$ has the area version of the invariant mean value property, must u be harmonic? This question has an affirmative answer if we replace the hypothesis that u is in $C(D) \cap L^1(D, dA)$ with the stronger hypothesis that $u$is in$C(\bar{D})$; see in [4,5].

We need to consider functions that are not necessarily continuous on the closed disk, so the result mentioned in the last sentence will not suffice. However, our functions do have the property that their radiation are continuous on the closed disk, and we will prove that this property, along with the area version of the invariant mean value property, is enough to imply harmonicity.

If$u \in C(D)$, then the radiation of u denoted$R(u)$, is the function on D defined by

$$R(u)(w)=\int_0^{2\pi} u\left(e^{i\theta}\right)\frac{d\theta}{2\pi} \qquad (3)$$

In the following lemma, which will be a key tool in our proof of Theorem 1, the statement $R(u^{\circ}h) \in C(\bar{D})$ means $R(u^{\circ}h)$ can be extended to a continuous complex valued function on $\bar{D}$

Lemma 2. Suppose that$u \in C(D) \cap L^1(D, dA)$. Then u is harmonic on D if and only if

$\int_D uoh \frac{dA}{\pi} = u\left(h(0)\right)$ (4)
and
$R(u^\circ h) \in C(\overline{D})$ for every$h \in Aut(D)$ (5)

Proof. Suppose that u is harmonic on$D$. Let$h \in Aut(D)$. As we discussed earlier, $u^\circ h$ is harmonic and we see in Eq(4) holds. The mean value property implies that $R(u^\circ h)$ is a constant function on$D$, with value$u\left(h(0)\right)$, we see in Eq (5) also holds.

To prove the other direction, suppose that Eq(4) and Eq(5) hold. Let$h \in Aut(D)$, and let

$$v \in R(u^\circ h)$$

from Eq(5) we fine $v \in C(\overline{D})$

We want to show that v has the area version of the invariant mean value property. To do this, fixg $\in Aut(D)$. Then

$\int_D u^\circ g \frac{dA}{\pi} u = \int_D \mathcal{R}(u^\circ h)\left(g(w)\right) \frac{dA(w)}{\pi} = \int_D \int_0^{2\pi} u\left(h\left(g(w)e^{i\theta}\right)\right) \frac{d\theta}{2\pi} \frac{dA(w)}{\pi}$ (6)

$\theta \in [0,2\pi]$ To check that interchanging the order of integration in the last integral is valid, for each define $f_\theta \in Aut(D)$ by

$$f_\theta(w) = h\left(g(w)e^{i\theta}\right)$$

The inverse f_θ^{-1} of f_θ is also an analytic automorphism of$D$, so there exist $\alpha \in \partial D$ and $\beta \in D$ such that $f_\theta^{-1}(z) = \frac{\beta - Z}{1 - \overline{\beta}_Z}$, for all $Z \in D$

Thus

$\left|\left(f_\theta^{-1}\right)^{/}(z)\right| = \frac{1-|\beta|^2}{\left|1-\overline{\beta}_z\right|^2} \leq \frac{1+|\beta|}{1-|\beta|}$, for all $Z \in D$

Note that$\beta = f_\theta(0) = h\left(g(0)e^{i\theta}\right)$; we are thinking of h and g as fixed, so the above inequality shows there is a constant K suchthat

$\left|\left(f_\theta^{-1}\right)^{/}(z)\right| \leq K$, for all $Z \in D$ and $\theta \in [0,2\pi]$

Now

$$\int_0^{2\pi} \int_D \left|u\left(h\left(g(w)e^{i\theta}\right)\right)\right| \frac{dA(w)}{\pi} \frac{d\theta}{2\pi} = \int_0^{2\pi} \int_D |u(z)| \left|\left(f_\theta^{-1}\right)^{/}(z)\right| \frac{dA(z)}{\pi} \frac{d\theta}{2\pi}$$
$$\leq K^2 \int_D |u(z)| \frac{dA(z)}{\pi} \leq \infty$$

That is apply Fubini's Theorem to Eq(6), getting

$$\int_D v^\circ g \frac{dA}{\pi} = \int_0^{2\pi} \int_D u\left(h\left(g(w)e^{i\theta}\right)\right) \frac{dA(w)}{\pi} \frac{d\theta}{2\pi} = \int_0^{2\pi} \int_D \left(v^\circ f_\theta(w)\right) \frac{dA(w)}{\pi} \frac{d\theta}{2\pi}$$
$$= \int_0^{2\pi} u\left(f_\theta(0)\right) \frac{d\theta}{2\pi} = \int_0^{2\pi} u\left(h\left(g(0)e^{i\theta}\right)\right) \frac{d\theta}{2\pi} = \mathcal{R}(u^\circ h)\left(g(0)\right) = v\left(g(0)\right)$$

Thus v is a continuous function on $\overline{D}$ that has the area version of the invariant mean value property. Hence v is harmonic on D [4,5]. Because v is also a radial function, the mean value property implies that v is a constant function on D, with value $v(0)$. Recall that$v = \mathcal{R}(u^\circ h)$, so

$$\int_0^{2\pi} (u^\circ h)\left(re^{i\theta}\right) \frac{d\theta}{2\pi} = u\left(h(0)\right)$$

for every $r \in [0,1)$ and for each $h \in Aut(D)$. In other words, u has the invariant mean value property. Thus in [4], u is harmonic on D.

As mentioned earlier, it is unknown whether Lemma 2 remains true if Eq(5) is deleted. We believe that the following proposition, which reduces this question to a tempting integral equation, is the best way to attack this problem. Patrick Ahern and Walter Rudin also independently proved Lemma 2 and Proposition 3 at about the same time we did.

Proposition 3. Suppose that the constant functions are the only functions $V \in C([0,1)) \cap L^{/}[0,1]$ such that

$v(t) = (1-t)^2 \int_0^1 \frac{1+ts}{(1-ts)^2} V(s)ds$, for every $t \in [0,1)$ (7)

Then every function in $C(D) \cap L^{/}(D, dA)$ having the area version of the invariant mean value property is harmonic.

Proof: First, suppose that v is a radial function in $C(D) \cap L^{/}(D, dA)$ having the area version of the invariant mean value property. We will show that v is constant on D. For $\alpha \in [0,1)$, let $h_\alpha \in Aut(D)$ be defined by

$h_\alpha(z) = \frac{\alpha - z}{1-\alpha z}$

Note that h_α is its own inverse under composition. For each $\alpha \in [0,1)$ we have

$$v(\alpha) = \int_D ((v^\circ h_\alpha))(z)\frac{dA(z)}{\pi} = \int_D v(w)\left|h^{/}_\alpha(w)\right|^2 \frac{dA(z)}{\pi}$$

$= (1-\alpha^2)^2 \int_0^1 v(r)\, r \int_0^{2\pi} \frac{1}{\left|1-\alpha r e^{i\theta}\right|^4}\frac{d\theta}{2\pi} dr$

$= (1-\alpha^2)^2 \int_0^1 \frac{r(1+a^2r^2)}{(1-a^2r^2)^3} v(r)dr$

$= (1-\alpha^2)^2 \int_0^1 \frac{1+a^2r^2}{(1-a^2s)^3} v(\sqrt{s})ds$

In the above equation, replace α with $\sqrt{t}$ and define a function V on [0, 1) by $V(t) = v(\sqrt{t})$, transforming the above equation into Eq(7). Hence V is constant on [0,1), and thus so is v, as claimed.

To complete the proof, now suppose that u is a function in $C(D) \cap L^{/}(D, dA)$, having the area version of the invariant mean value property. Let $h \in Aut(D)$. Clearly, $\mathcal{R}(u^\circ h)$ is a radial function on D, and, as shown in the proof of Lemma 2, hasthe area version of the invariant mean value property. By the above paragraph, $\mathcal{R}(u^\circ h)$ is constant on D. In particular, $\mathcal{R}(u^\circ h) \in C(\overline{D})$, and so by Lemma 2, u is harmonic.

3. The Toeplitz Operators

For $h \in Aut(D)$, define an operator U_h on L_a^2 by

$U_h f = (u^\circ h)h$

A simple computation shows that U_h is a unitary operator from L_a^2 onto L_a^2, with inverse U_{h-1}.

In the following lemma that proof of Theorem 1.

Lemma 4. Let $h \in Aut(D)$ and let $(\varphi + 1) \in L^\infty(D, dA)$. Then

$U_h T_{\varphi+1} U_h^* = T_{(\varphi+1)^\circ h}$

Proof. Define $V_h: L^2(D, dA) \to L^2(D, dA)$ by $V_h f = (f^\circ h)h'$. Then V_h is a unitary operator from $L^2(D, dA)$ onto $L^2(D, dA)$. Obviously $V_h L_a^2 = U_h$. Thus V_h maps L_a^2 onto L_a^2, so

$PV_h = V_h P$ (8)

If $f \in L_a^2$, so that

$$T_{(\varphi+1)^\circ h} U_h f = T_{(\varphi+1)^\circ h}(f^\circ h)h' = P\left[\left((\varphi + 1)^\circ h\right)(f^\circ h)h'\right]$$
$$= P\left[V_h\left((\varphi + 1)f\right)\right] = V_h\left[P\left((\varphi + 1)f\right)\right] = U_h T_{\varphi+1} f$$

Thus $T_{(\varphi+1)^\circ h} U_h = U_h T_{\varphi+1}$ and because U_h is unitary, this implies the desired result.

Let $H^{\varphi+1}(D)$ denote the usual is the Hardy space on the disk. It is well known that $H^1(D) \subset L_a^2$. In the proof of Theorem 1 we will use, without comment, the following consequence: If $f, g \in H^2(D)$, then $f, g \in L_a^2$, and thus $f\bar{g} \in L^2(D, dA)$.

We have now assembled all the ingredients needed to prove Theorem 1.

Proof of theorem 1. If we begin with the easy direction. First suppose that (a) holds, so that $(\varphi + 1)$ and $(\psi + 1)$ are analytic on D which means that $T_{\varphi+1}$ and $T_{\psi+1}$ are, respectively, the operators on L_a^2 of multiplication by $(\varphi + 1)$ and $(\psi + 1)$. So that$T_{\varphi+1}T_{\psi+1} = T_{\psi+1}T_{\varphi+1}$.

Now suppose that (b) holds, so that $\overline{\varphi + 1}$ and $\overline{\psi + 1}$ are analytic on$D$. By the paragraph above, $T_{\overline{\varphi+1}}T_{\overline{\psi+1}} = T_{\overline{\psi+1}}T_{\overline{\varphi+1}}$.

Take the adjoin of both sides of this equation, and use the identity $T^{*}_{\overline{\varphi+1}} = T_{\varphi+1}$ to conclude that$T_{\varphi+1}T_{\psi+1} = T_{\psi+1}T_{\varphi+1}$.

Finally suppose that (c) holds, so there exist constants$a, b \in C$, not both 0, such that $a(\varphi + 1) + b(\psi + 1)$ is constant on$D$. If$a \neq 0$, then there exist constants $\beta, \gamma \in C$ such that $\varphi + 1 = \beta(\psi + 1) + \gamma$ on$D$, which means that $T_{\varphi+1} = \beta T_{\psi+1} + \gamma I$ (I denotes the identity operator on L^2_a), which clearly implies that $T_{\varphi+1}T_{\psi+1} = T_{\psi+1}T_{\varphi+1}$. If $b \neq 0$, we conclude in a similar fashion that $T_{\varphi+1}$ and $T_{\psi+1}$ commute.

Now to prove the other direction of Theorem 1, suppose that $(\varphi + 1)$ and $(\psi + 1)$ are bounded harmonic functions on D such that$T_{\varphi+1}T_{\psi+1} = T_{\psi+1}T_{\varphi+1}$. Because $\varphi + 1$ and $\psi + 1$ are harmonic on$D$, there exist functions f_1, f_2, g_1 and g_2 are analytic on D such that t

$$(\varphi + 1) = f_1 + \overline{f_2} \text{ and } (\psi + 1) = g_1 + \overline{g_2} \text{ on } D \tag{9}$$

Because $(\varphi + 1)$ and $\psi + 1$ are bounded on D, the functions f_1, f_2, g_1 and g_2 must be in $H^2(D)$.

This means that the function is constant at D. So

$$\begin{aligned} T_{\varphi+1}T_{\psi+1}1 &= T_{\varphi+1}\big(P(\psi + 1)\big) = T_{\varphi+1}\big(P(g_1 + \overline{g_2})\big) = T_{\varphi+1}\big(g_1 + \overline{g_2(0)}\big) \\ &= P\big([f_1 + \overline{f_2}][g_1 + \overline{g_2(0)}]\big) = f_1g_1 + \overline{g_2(0)}f_1 + P\big(\overline{f_2}g_1\big) + \overline{f_2(0)g_2(0)} \\ &= \langle T_{\varphi+1}T_{\psi+1}1,1\rangle = \langle f_1g_1 + \overline{g_2(0)}f_1 + \overline{f_2}g_1 + \overline{f_2(0)g_2(0)}, 1\rangle \\ &= \int_D \big(f_1g_1 + \overline{g_2(0)}f_1 + \overline{f_2}g_1 + \overline{f_2(0)g_2(0)}\big)dA \\ &= \pi\big[f_2(0)g_1(0) + f_1(0)\overline{g_2(0)} + \overline{f_2(0)g_2(0)}\big] + \int_D \overline{f_2}g_1 dA \end{aligned} \tag{10}$$

A sirnilar formula can be obtained fo $\langle T_{\varphi+1}T_{\psi+1}1,1\rangle$

Because $T_{\varphi+1}T_{\psi+1} = T_{\psi+1}T_{\varphi+1}$ we can set the right-hand side of Eq (10) equal to the corresponding formula for $\langle T_{\varphi+1}T_{\psi+1}1,1\rangle$,getting

$$\int_D \big(\overline{f_2}g_1 - f_1\overline{g_2}\big)\frac{dA}{\pi} = \overline{f_2(0)}g_1(0) - f_1(0)\overline{g_2(0)} \tag{11}$$

Let $h \in Aut(D)$. Multiplying both sides of the equation $T_{\varphi+1}T_{\psi+1} = T_{\psi+1}T_{\varphi+1}$ on the left and by U_h^* on the right, and recalling that U_h is unitary so that$U_h^*U_h = 1$, we get

$$U_hT_{\varphi+1}U_h^*U_hT_{\psi+1}U_h^* = U_hT_{\psi+1}U_h^*U_hT_{\varphi+1}U_h^*$$

Lemma 4 now shows that

$$T_{(\varphi+1)*h}T_{(\psi+1)*h} = T_{(\psi+1)*h}T_{(\varphi+1)*h} \tag{12}$$

Composing both sides of the equations in Eq(8) with h expresses each of the bounded harmonic functions $(\varphi + 1) * h$ and $(\psi + 1) * h$ as the sum of an analytic function and a conjugate analytic function:

$$(\varphi + 1) * h = f_1 * h + \overline{f_2} * h \text{ and } (\psi + 1) * h = g_1 * h + \overline{g_2} * h \text{ on } D \tag{13}$$

From Eq(11) was derived under the assumption that $T_{\varphi+1}T_{\psi+1} = T_{\psi+1}T_{\varphi+1}$;thus Eq(12), combined with Eq(14), says that Eq(11) is still valid when we replace each function in it by its composition with h. In other words ,

$$\int_D \big(\overline{f_2}g_1 - f_1\overline{g_2}\big) * h\frac{dA}{\pi} = \overline{f_2}\big(h(0)\big)g_1\big(h(0)\big) - f_1\big(h(0)\big)\overline{g_2}\big(h(0)\big)$$

Letting

$$u = \overline{f_2}g_1 - f_1\overline{g_2}$$

the equation above becomes

$$\int_D u * h\frac{dA}{\pi} = u\big(h(0)\big)$$

In the other words, u has the area version of the invariant mean value property.

We can want to show that u is harmonic on$D$. By the above equation and Lemma 2, we need only show that$\mathcal{R}(u * h) \in C(\overline{D})$. To do this, represent the analytic functions $f_2 * h$ and $g_1 * h$ as Taylor series:

$$(f_2 * h)(z) = \sum_{n=0}^{\infty} \alpha_n Z^n \ \ and\ (g_1 * h)(z) \sum_{n=0}^{\infty} \beta_n Z^n, \qquad for\ all\ Z \in D$$

Because $(\varphi + 1) * h$ and $(\psi + 1) * h$ are bounded harmonic functions on$D$, Eq(13) implies that functions $f_2 * h$ and $g_1 * h$ are in $H^2(D)$ $H2(D)$, so that

$$\sum_{n=0}^{\infty} |\alpha_n|^2 < \infty \ \ and\ \sum_{n=0}^{\infty} |\beta_n|^2 < \infty \qquad (14)$$

Now for $Z \in D$ we have

$$\mathcal{R}\left(\left(\overline{f_2} g_1\right) * h\right)(z) = \int_0^{2\pi} \left(\overline{f_2} * h\right)\left(ze^{i\theta}\right)(g_1 * h)\left(ze^{i\theta}\right)\frac{d\theta}{2\pi} = \sum_{n=0}^{\infty} \overline{\alpha_n}\beta_n |Z|^{2n}$$

The inequalities in Eq(14) imply that $\sum_{n=0}^{\infty} |\alpha_n \beta_n| < \infty$, so the above formula for $\mathcal{R}\left(\left(\overline{f_2} g_1\right) * h\right) \in C(D)$ shows that $\mathcal{R}\left(\left(\overline{f_2} g_1\right) * h\right) \in C(\overline{D})$; similarly, we get that $\mathcal{R}\left(\left(f_1 \overline{g_2}\right) * h\right) \in C(\overline{D})$. So that$\mathcal{R}(u * h) \in C(\overline{D})$, as desired. Thus at this stage of the proof we know that u is harmonic.

Let $\frac{\partial}{\partial z}$ and the $\frac{\partial}{\partial z}$ denote the usual operators defined by

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)\ and\ \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$$

If f is analytic, then the Cauchy-Riemann equations show that

$$\frac{\partial f}{\partial z} = f, \quad \frac{\partial f}{\partial \bar{z}} = 0, \ \frac{\partial \bar{f}}{\partial z} = 0, and\ \frac{\partial \bar{f}}{\partial z} = \bar{f}.$$

It is easy to check that $\frac{\partial}{\partial z}$ and $\frac{\partial}{\partial z}$ obey the usual addition and multiplication formulas for derivatives and that

$$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = 4\frac{\partial}{\partial \bar{z}}\frac{\partial}{\partial z}$$

Thus, because u is harmonic, we have

$$0 = 4\frac{\partial}{\partial \bar{z}}\left(\frac{\partial u}{\partial z}\right) = 4\frac{\partial}{\partial \bar{z}}\left|\frac{\partial\left(\overline{f_2} g_1 - f_1 \overline{g_2}\right)}{\partial z}\right| = 4\frac{\partial}{\partial \bar{z}}\left(\overline{f_2} g_1' - f_1' \overline{g_2}\right) = 4\left(\overline{f_2} g_1' - f_1' \overline{g_2}\right)$$

Hence

$\overline{f_1} g_2' - f_2' \overline{g_1}$ (15)

We finish the proof by showing that the above equation implies that (a), (b), or (c)

Holds. If g_1' is identically 0 on$D$, then Eq(15) shows that either g_2' is identically 0 on D, then $(\psi + 1)$ would be constant on D and (c) would hold or f_1' is identically 0 on D, so both $(\varphi + 1)$ and $(\psi + 1)$ would be analytic on D and (b) would hold. Similarly, if g_2' is identically 0 on$D$, then Eq(15) shows that either (c) or (a) would hold. Thus, we may assume that neither g_1'nor $g_2'$is identically 0 on$D$, and so Eq (15) shows that at all points of D except the countable set consisting of the zeroes of $g_1' g_2'$.

$$\frac{f_1'}{g_1'} = \left[\frac{f_2'}{g_2'}\right]^{-}$$

The left-hand side of the above equation is an analytic function on D with the zeroes of$g_1' g_2'$, deleted, and the right hand side is the complex conjugate of an analytic function on the same domain, and so both sides must equal a constant $c \in C$. Thus$f_1' = c g_1'$, and $f_2' = \bar{c} g_2'$ on$D$. Hence $f_1 - c g_1$ and $\overline{f_2} - c \bar{g}_2$ are constant on D, and so their sum, which equals $(\varphi + 1) - c(\psi + 1)$, is constant on D; in other words, (c) holds and the proof of Theorem 1 is complete.

Recall that an operator is called normal if it commutes with its adjoint. We can use Theorem1 to prove the following corollary, which states that for $(\varphi + 1)$ a bounded harmonic function on$D$, the Toeplitz operator $T_{\varphi + 1}$is normal only in the obvious case.

Corollary 5. Suppose that $(\varphi + 1)$ is a bounded harmonic function on$D$. Then $T_{\varphi + 1}$ is a normal operator if and only if $(\varphi + 1)(D)$ lies on some line in$C$.

Proof: First, suppose that $(\varphi + 1)(D)$ lies on some line in$C$.

Then there exist constants$\alpha, \beta \in C$, with$\alpha \neq 0$, such that$\alpha(\varphi + 1) + \beta$, is real valued on$D$. So that $T_{\alpha(\varphi + 1)+\beta}$ is a self-adjoint operator, and hence $T_{\varphi + 1}$ which equals$\alpha^{-1}T_{\alpha(\varphi + 1)+\beta-\beta I}$, is a normal operator.

To prove the other direction, suppose now that $T_{\varphi + 1}$ is a normal operator. So that $T_{\varphi + 1}T_{\overline{\varphi + 1}} = T_{\overline{\varphi + 1}}T_{\varphi + 1}$ and so Theorem 1 implies that $(\varphi + 1)$ and $\overline{(\varphi + 1)}$ are both analytic on D (in which case $(\varphi + 1)$ is constant, so we are done) or there are Constants$\alpha, \beta \in C$, not both 0, such that $a(\varphi + 1) + b\overline{(\varphi + 1)}$ $(\varphi + 1)$ is constant on D. The latter condition implies that $(\varphi + 1)(D)$ lies on a line.

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