Edelweiss Applied Science and Technology
ISSN: 2576-8484
Vol. 8, No. 6, 3693-3703
2024
Publisher: Learning Gate
DOI: 10.55214/25768484.v8i6.2810

Comprehending geometry proofs: A study on prospective mathematics teachers' perceptions and preferences of proof formats

Lathiful Anwar[1*], Cholis Sa'dijah[2], Tomi Listiawan[3], Anita Dewi Utami[4], Hutkemri Zulnaidi[5], Leong Kwan Eu[6]
[1,2,3,4]Department of Mathematics, Universitas Negeri Malang, Indonesia; lathiful.anwar.fmipa@um.ac.id (L.A.).
[5,6]Department of Mathematics and Science Education, Universiti Malaya, Malaysia.

Abstract: The explicative aim of this research is to investigate the optics of proof format preferences held by prospective mathematics teachers (PMTs), in an effort to assess the effect that such preferences may have on their understanding of geometric proofs. The researchers employed a qualitative case study approach, in which six PMTs participated in Focus Group Discussions (FGDs) on the three styles of proof: paragraph proof, two-column proof and flow-chart proof. The methodology employed also focused on the formatting models by exploring the participants' suggestions regarding each format in order to determine their cognitive responses as well as their preferences. The study exposes different preferences among PMTs which are a result of their personal cognitive style and experience with proofs. Some of the PMTs preferred the two-column format due to its systematic and logical nature, other PMTs preferred the storytelling nature of the paragraph proof format sponsored also flow-chart representation. These preferences bring out the requirement for the development of pedagogical strategies that employ different proof formats in order to meet the needs of different types of learners for effective understanding. In summary, this research demonstrates the need for customizing proof instruction to different students' needs which can hopefully enhance PMTs' responses and interactions with the concept of mathematical proofs. The practical outcomes show that teachers should not be afraid to think in more flexible witnessing forms of proof in order to aid students learning while minimizing the cognitive load in line with the principles of constructivist learning. It would be worthwhile however to extend this research and examine what these preferences would mean for the future and how those would fit within the context of online education with the overall aim of creating a better mathematics curriculum that is more effective and has a wider reach.
Keywords: *Comprehending proof, Geometry proof, Preference, Proof formats.*

1. Introduction

Proofs serve as the fundamental manifestation of mathematical verification, encompassing the core principles of mathematical exploration and comprehension [1]–[3]. Proofs are considered essential in the field of mathematics as they provide conclusive evidence of mathematical facts. Moreover, they play a crucial role in education by aiding in the cultivation of mathematical thinking, reasoning, and literacy, as stated by the National Council of Teachers of Mathematics [4]. The act of actively participating in the process of proving mathematical theorems enables students to explore and analyze the complexities inherent in mathematical concepts, fostering a deeper understanding and recognition of the logical consistency and interconnectedness of the subject matter [5].

The importance of proof in the practice of education is especially acknowledged in connection with the study of mathematics, since it should help students to think and reason at a higher and more complex level [6]. The act of proving goes beyond simple verification and assumes a role of rewriting history by transforming students into mathematicians in measurement—they are able to do more than only consume mathematical knowledge; they produce it [7]. Notwithstanding their pedagogical role,

* Correspondence: lathiful.anwar.fmipa@um.ac.id

nevertheless, the teaching and learning of proofs remains a major source of concern. In that regard, [8] reports that there are students who understand and appreciate the proofs provided due to the approach taken to present them. Because of this observation, the tendency to systematically employ various forms of proofs in teaching so as to suit the various students' learning and cognitive styles has been on the increase [2].

Specifically, the research findings indicated that paragraph, two-column, and flow-chart proof explanations fulfill different proof reading learning and cognitive needs [9]. For instance, it is possible that students who have great difficulties in reading find the narrative logic in paragraph proofs cumbersome and preferred the strict format of two-column or the picture flow chart instead [2]. Further studies reveal that in particular the use of flowchart diagrams can assist the reasoning process as it helps to formulate logical steps in a sequential order [2].

The difference in the success of teaching methodologies and specifically proof formats calls for an urgent need to comprehend more adequately how learners understand and use geometrical proofs. Investigating students' pointing out preferences in proof geometrical formats and giving the explanation of the reasons and experiences for using each format may help understand how best to teach and learn proof [10]. It suffices to say that the above mentioned results are quite important regarding the tailoring of the teaching approaches to the needs of individual learners and the improvement of mathematics education in general.

Despite the growing body of literature on proof instruction, there is a significant gap in understanding how prospective mathematics teachers (PMTs) engage with various proof formats and how these formats influence their learning and teaching preferences. Given the pivotal role that future educators play in shaping mathematical education, it is crucial to explore how their preferences toward proof formats may impact their instructional practices and the mathematical competencies of their students.

This study addresses the following research questions: (1) How do PMTs perceive the effectiveness of different proof formats (paragraph, two-column, and flow-chart) in understanding geometrical proofs? (2) What cognitive factors influence PMTs' preferences for specific proof formats? And (3) How do PMTs' proof format preferences influence their potential future teaching strategies? Considering such qualitative case study methods as PMT hypotheses, the study presents the possible consequences that the Proof Format Preference Theory might have for pedagogy. The stronger the relationship between students' cognitive styles and proof format, the better the curriculum design can address students' varying learning needs. This kind of research helps not only to fill in gaps in theory related to cognitive load along with constructivist learning theories but also further to provide some recommendations useful for improving proof instruction in mathematics education. While concentrating on PMTs, this research builds on previous studies concerning proof education and the role of teachers in its process. They are invited to discuss what they do and prefer when teaching or learning proofs with a view to devising means of enhancing proofs in order to create better (inter)national mathematical educational standards.

2. Research Method

2.1. Research Design

The research scope of the study follows qualitative research paradigm by employing a case study method which is an interesting way of examining and understanding phenomena within their real settings [11]. This methodology is particularly suitable for understanding the complex perspectives and attitudes of prospective mathematics teachers (PMTs) toward using various formats of proof. The application of case studies makes it possible to consider the wide range of individual cases and is known to be useful in explaining the complicated nature of educational practice [12], [13].

2.2. Participants

In this study, a group of six prospective mathematics teachers (PMTs) was selected purposely and with consideration of gender balance in this case an equal number of male and female participants were involved. The consideration of gender in this aspect was made in order to see if there could be any likely cognitive or perceptual differentiations in the way they interact with the proof formats. The purpose of the research was not about investigating gender differences but this element is important in anticipation of any gender-related trends or insights that may arise from the data in an internally consistent manner [14].

The participants were selected based on specific criteria, including prior exposure to the subject of triangle congruence and familiarity with two-column proofs. These shared experiences helped establish a homogeneous group, thereby fostering meaningful dialogue during focus group discussions [15]. Despite the small sample size, the gender balance aimed to represent diverse perspectives within the group, acknowledging the importance of considering potential individual differences in future studies.

2.3. Data Collection

PMTs participated in a guided inquiry where they responded to carefully crafted questions aimed at probing their comprehension with the given proofs. The Figure 1 depicts the given statement and the proof presented in three formats: paragraph, two-column and flow-chart proof in web-based environment developed by the authors, see Figure 1. After reading all proof formats, they chose one of formats to be read in order to answer the questions aiming to assess their reading comprehension. The reading comprehension test was adopted from the study by [2]. The questions were formulated to elicit cognitive and affective responses, covering both the understanding of content and the personal experiences of the PMTs with each format [16]. Subsequently, discussions were steered towards more open-ended inquiry, prompting PMTs to discuss their format preferences the rationale behind their choices, and to critically evaluate the merits and drawbacks of each format. The rich dialogue from these discussions was recorded and transcribed, providing a substantive body of data for analysis [17].

As shown in the figure 1, if L, the perpendicular bisector of $\overline{BC}$, intersects $\overline{AB}$ at D, and intersects $\overline{BC}$ at M; and $\overline{DA} \cong \overline{DB}$; then $\angle DCA \cong \angle DAC$

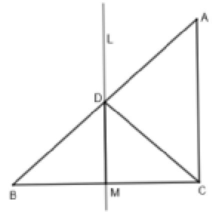


Figure 1.

A Paragraph Proof:

As shown in the figure,

Since L, the perpendicular bisector of $\overline{BC}$, intersects $\overline{BC}$ at M ... Line 1

$m\angle BMD = m\angle CMD = 90$ and $\overline{BM} \cong \overline{CM}$ Line 2 .... line 2

And $\overline{DM} \cong \overline{DM}$ (axiom of reflective of congruency), ... Line 3

$\therefore \Delta BMD \cong \Delta CMD$ (axiom S-A-S) .... Line 4

$\therefore \overline{DB} \cong \overline{DC}$ (def. of congruent polygons, correspond sides)........ Line 5

And $\overline{DA} \cong \overline{DB}$, ... Line 6

From Line 5 and Line 6 → $\overline{DA} \cong \overline{DC}$. .... Line 7

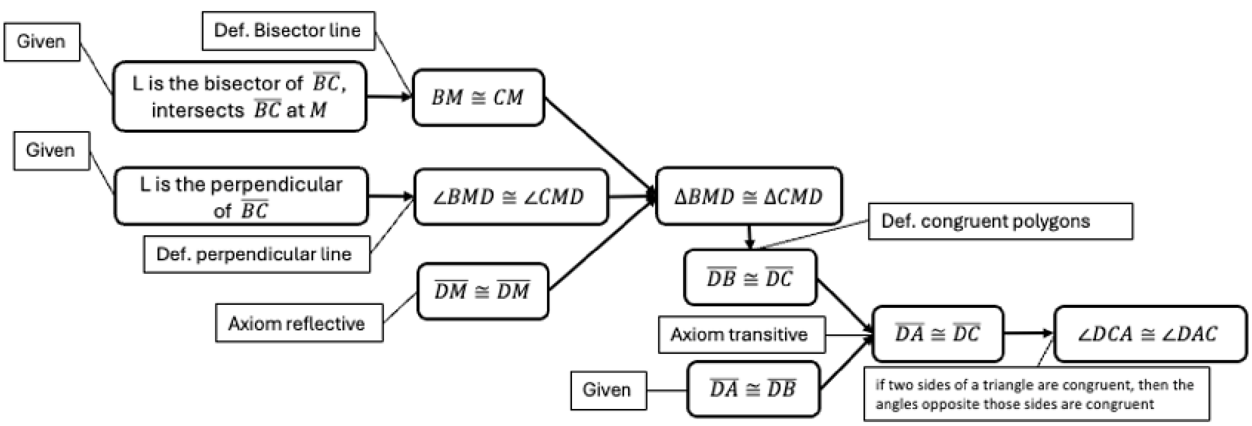
Because $\overline{DA} \cong \overline{DC}, \angle DCA \cong \overline{DAC}$.Line....................................... Line 8

1(a) Paragraph proof

A Two-column Proof:

Statements	Reasons
L, the perpendicular bisector of $\overline{BC}$, intersects $\overline{BC}$ at M	Given
$m\angle BMD = m\angle CMD = 90$ and $\overline{BM} \cong \overline{CM}$	Definition of perpendicular lines and definition of bisector of line segment
$\overline{DM} \cong \overline{DM}$	Axiom of reflective of congruency
$\Delta BMD \cong \Delta CMD$	Axiom S-A-S
$\overline{DB} \cong \overline{DC}$	Definition of congruent polygons
L intersects $\overline{AB}$ at D and $\overline{DA} \cong \overline{DB}$	Given
$\overline{DA} \cong \overline{DC}$	Axiom of transitive of congruency
$\angle DCA \cong \overline{DAC}$	Theorem 9: if two sides of a triangle are congruent, then the angles opposite those sides are congruent

1(a) Two-column proof



1(c) Flow-chart proof

Figure 1.
There are three option of proof format to be chosen by PMTs to comprehend the proof.

2.4. Data Analysis

Thematic analysis, a technique employed to uncover, analyze, and present patterns within data [18], was utilized to examine the transcribed FGDs. The methodology employed in this study encompassed a recursive procedure of encoding data in an inductive fashion, systematically identifying emerging themes, and subsequently building a comprehensive interpretation of the data in relation to the research objectives [19].

2.5. Ethical Considerations

All procedures performed in studies involving human participants were in accordance with the ethical standards of the institutional and/or national research committee and with the 1964 Helsinki Declaration and its later amendments or comparable ethical standards. Informed consent was obtained from all individual participants included in the study. To ensure confidentiality, all PMTs were assigned pseudonyms throughout the study (American Educational Research Association [20].

3. Findings and Discussion

This qualitative case study provides a comprehensive examination of the preferences for format among six prospective mathematics teachers (PMTs) as they interact with geometrical proofs, which are fundamental in the field of mathematical education and cognition. Each prospective mathematics teacher (PMT), labeled as S1 through S6, offers a distinct viewpoint that enhances our comprehension of how the proof format impacts geometrical proof comprehension.

Within the context of our research findings, it is noteworthy to mention that Participant S1 emerges as an anomalous case, displaying a distinct preference for employing the paragraph form when reading and presenting geometrical proofs. S1 articulates that the continuity and storytelling nature of the narrative paragraph form a coherent thread that unravels the complexities of mathematical proofs with greater clarity, see the transcript 1 (turn 2 and 6). The aforementioned preference is not an independent occurrence; it aligns with the findings of the research conducted by [21], who proposed that narrative structures within the field of mathematics possess significant potential for providing extensive contextualization and facilitating a deeper understanding of concepts. These aspects are of utmost importance when maneuvering through the complex intricacies of mathematical proofs.

Interestingly, S1 (male) favored the paragraph format, aligning with studies that have shown a male propensity for narrative structures that allow for a sequential understanding of concepts [22]. This finding could be connected to the cognitive styles typically attributed to males, which often involve a preference for step-by-step logical structuring of information [23].

Table 1.

Turn	Transcript	
1	R	: Now I want to know, I want to hear the testimonials. First, why were this format chosen by you as the most pleasant to read, to understand, to create? Let's start from the left.
2	S1	: Why I chose this format, the Paragraph, because it's like more detailed.
3	R	: More detailed? Is there a story, how do you see the details? Is the detail seen in how the statements are related?
4	S1	: I use more words.
5	R	: Okay, so the proof must be in narrative form, right?
6	S1	: Yes, narrative. That is, after this, then that, because of this, it leads to that, right? So the narrative of how the statements are connected needs to be explicit.

In contrast, S2 and S3 demonstrate a shared preference for the utilization of the two-column format. The inclination described can be traced back to the individual's experience during the class activity, where the major instructional method employed was the two-column method, see transcript 2 (turn 4 and 22). This method is characterized by its plain and systematic approach, meticulously pairing each mathematical statement with its matching justification or reason, see transcript 2 (turn 6 – 10, 22 and 26). The preference for the two-column style is supported by the findings of [24], who recognize its effectiveness in enhancing the acquisition of both procedural and conceptual knowledge.

Table 2.

Turn	Transcript	
1	R	: What about the other format from earlier? What's your name? Nickname?
2	S2	: Tabel
3	R	: Why Tabel?
4	S2	: Because I usually study using tables.
5	R	: Usually with tables. What do you think is good about writing in a table?
6	S2	: Because it's nice, step 1 because the reason is this, step 2 because the reason is this.
7	R	: Oh, so the connection between why this statement can be written and its explicit reason is right next to it?

8	S2	: Yes.
9	R	: This appears, this can appear because of this, right?
10	S2	: Yes. So, the relationship between the statement and its justification is clearly visible.
11	R	: The relationship can be clearly seen. This statement I can write because this guarantees that, right? And what else?
12	S2	: Not complicated [complex].
13	R	: You mean it's to the point. Why not a paragraph?
14	S2	: If it's a paragraph, it's long and complicated. So, you need to read it for a long time.
15	R	: Not complicated. Which statement is caused by what, yes? Then the reasons are already grouped. Anything else
16	S2	: It's neat.
17	R	: Neat. If you care more about the appearance.
18	S2	: Neat. Neatness makes it more understandable.
19	R	: More understandable. Why is that?
20	S2	: Because it's simpler.
21	R	: What about you? Did you also choose the table?
22	S3	: Yes. To make it more visible. It captures this. It's simpler.
23	R	: What's the simplicity of the table?
24	S3	: It's structured.
25	R	: Okay.
26	S3	: Between each claim and reason, the reason.

A notable occurrence arises when S4, S5, and S6 are confronted with the flow-chart structure. The three prospective mathematics teachers (PMTs) are visually represented in a manner that effectively illustrates logical relationships, with arrows skillfully delineating the linkages between statements, see transcript 3 (turn 2, 8, 10, 12, 37 and 43). Utilizing a visual-spatial arrangement enhances the ease of navigating through mathematical proofs, supporting the claims made by [25] that visual representations can greatly simplify the comprehension of subtle linkages within complex information. The transition of preference from the two-column to the flow-chart format for S4, S5, and S6 could be indicative of a visual-spatial learning style. This style emphasizes the importance of seeing information in order to process it effectively [26]. Flow-charts offer a graphical representation of information flow and relationships, which can be more conducive to those who comprehend and retain information better through visual means.

Table 3.

Turn	Transcript	
1	R	: based on the comparasion between the three formats, which format you prefere to read in order to understand them?
2	S4	: Flow-chart.
3	R	: Why didn't you choose a flow-chart before?
4	S4	: Because I've never seen one before.
5	R	: You were skeptical earlier, right?
6	S4	: Nods and smiles
7	R	: Why is the flow-chart superior?
8	S4	: In our view, it's more detailed because it has directional arrows.
9	R	: What about the arrows? Can you really tell anything from them?
10	S4	: You know the process flow.
11	R	: How is the flow?
12	S5	: You can understand the future flow.

13	R	: What else is an advantage of flow-charts compared to two-column evidence?
14	S4	: The reasons are also visible.
15	R	: Do you mean that we can also track the reasons?
16	S5, S4, S3	: Yes.
17	R	: What are the disadvantages of flow-charts?
18	S2	: They are hard to create.
19	R	: Why is that?
20	S2	: With tables, you just write two columns straight down. With this, it's like guessing, it requires a lot of space.
21	R	: So, you mean the structure of two-columns is more default. Just straight down like that. But flow-charts seem never-ending?
22	S5, S4	: Yes (laughter).
23	R	: Okay, so that unpredictability can happen. Is this a lengthy process? If the advantage is just the subsequent language, maybe that's from the writing perspective. What else? What was complicated? The length you mentioned before, right? What else? Generally, how are things written in books?
24	S5	: Tables.
25	S1	: Words.
26	R	: Maybe that's what's considered formal evidence writing, that evidence is usually written in two columns or paragraphs. Why aren't paragraphs popular?
27	S6	: They're long. Unclear.
28	R	: Long. You have to create a narrative. What else is uncomfortable about paragraphs?
29	S5	: They're less tidy.
30	R	: Why are they untidy?
31	S6	: I prefer symbols
32	R	: Now, faced with the choice of evidence format, which one do you prefer?
33	S3, S4, S5	: Tables.
34	R	: If you're presented with proof again, in which format would you feel more comfortable reading it?
35	S3	: I like the table format, but for clarity, I like flow-charts, although they take a long time to create.
36	R	: If you were asked to read
37	S3, S4, S5, S6	: Flow-chart is easier to read.
38	R	: But when writing?
39	S4, S5	: Tables are easier.
40	R	: From what perspective?
41	S5	: In terms of space, practicality.
42	R	: Now, if it were for others to read...Intentionally, you want the reader to understand better. You would recommend or choose which format, so that the reader can grasp the meaning of your evidence without feeling tired due to limitations of space and time. If that's the intention, which would you choose?

43	S3, S4, S5, S6	: Flow-chart

In the context of gender, S2 and S3, also male, along with their female counterparts S4, S5, and S6, preferred the two-column format initially. This might indicate a gender-neutral perception of the effectiveness of structured, logical formats in mathematics learning, which aligns with the gender similarities hypothesis in cognitive abilities [14]. However, after the introduction of the flow-chart format, S4, S5, and S6 (all female) showed a preference for this visually oriented format. Previous studies have suggested that women may exhibit a greater inclination towards visual-spatial representations, as they can provide holistic overviews of the task at hand [27]. This may reflect a gendered response to the cognitive load associated with mathematical tasks, where the organization of information through visual means can reduce the intrinsic cognitive load for female learners [28].

Despite acknowledging the advantages of flow charts, S2 and S3 demonstrate a preference for the two-column format in terms of writing proofs due to its perceived efficiency and straightforwardness, see transcript 4 (turn 2, 4, and 6). Nevertheless, it is important to note that flow-charts provide a unique advantage in terms of communication due to their ability to visually depict mathematical relationships. This observation is consistent with [29] emphasis on the educational effectiveness of visual representations in the field of mathematics. The dichotomy in preferences when switching contexts, from personal understanding to communication (S2 and S3 for two-column, but flow-chart for teaching), suggests an adaptability that could be crucial for future educators. While there is a need for more research in this area, the pattern observed among the male participants aligns with research suggesting that males may choose efficiency over detail when under cognitive load but revert to more comprehensive methods when the task requires teaching or communication [30].

Table 4.

Turn	Transcript	
1	R	: If you were asked to present evidence in paragraph format, what would you do? Would you directly write the paragraph, or create a flow-chart or table first, then convert it into a paragraph? It seems that between the three, the paragraph is less popular. Now, if the final product is a paragraph and you're not time-constrained, which strategy would you choose? Would you narrate directly for the paragraph, or prepare a two-column table first, then narrate based on that, or draft a flow-chart first, then create the story in narrative? Assuming you're not tired and not time-constrained, which means it's easier to create evidence with more comfort and clarity?
2	S3	: I'd choose a table because it's easier to convert into a paragraph.
3	R	: Why not a flow-chart?
4	S3	: Because it's hard to convert it into a paragraph.
5	R	: So the easiest way to create a paragraph is from a table. But when reading evidence, you prefer?
6	S3, S4, S5, S6	: Flow-chart.

S1, maintaining a consistent stance, reiterates their preference for the paragraph format, finding it not only easier for writing proofs but also for explicating them, see transcript 3 (turn 25). This affinity towards narrative is reflective of [31] statements, which recognized a significant proclivity among mathematics teachers for narratives due to their inherent capacity to weave a more comprehensive understanding of mathematical concepts through storytelling. When considering learning styles, the preference for different formats seems to mirror the participants' underlying learning preferences. S1's choice of a narrative, paragraph format suggests a verbal or linguistic learning style, where language is the primary medium for understanding and expression [32]. The individual learning preference for narratives can cater to learners who excel in organizing information in a linear, textual format, typically characteristic of verbal learners.

These variances in format preferences underscore a critical facet of mathematics education: the importance of recognizing and embracing individual differences in learning styles. Such differences are vital considerations that should be at the forefront of instructional design, echoing [33] call for a pedagogical approach that is cognizant of these diverse learning needs.

In synthesizing these insights, the study reveals that the PMTs' preferences are not merely about comfort or familiarity; they represent a delicate balance between clarity, efficacy, and communicative power. The complexity of these preferences suggests a need for educators to diversify the formats of proof presentation to address the varied cognitive and communicative demands of their learners. This echoes [2] assertion that integrating multiple formats not only accommodates diverse learning preferences but also fosters a more holistic understanding of mathematics.

The findings of this case study shed insight into the complex relationship between individual learning preferences and the pedagogical practices that are most effective in facilitating them. The necessity for employing a comprehensive strategy in presenting mathematical proofs is apparent. This entails integrating many instructional components, such as narrative, structural, and visual features, in order to accommodate the broad array of cognitive processes. This method acknowledges and respects learners' unique characteristics and abilities and enhances the overall mathematical learning experience, leading to a stronger and more enduring understanding and enjoyment of mathematics.

4. Conclusion

This qualitative case study adds to the limited literature that explores the choices and experiences of prospective mathematics teachers (PMTs) in regards to the use of proof formats - paragraph, two-column, and flow-chart. It was established that PMTs' choices are mainly shaped by cognitive styles and their prior exposure to different proof formats as well as the context in which they expect to use the proofs.

With respect to the first research question, the investigation pointed out that the PMTs in the study viewed proof formats differently depending on the nature of their cognitive processes. Some participants (e.g., S1) still pointed out that while they favor flow charts, they welcome it last since it is almost too opening; a sticky reason is that it shifts away from a narrative flow they admit to craving even in pareto-prime. The simplicity in the two-column format allows for the linking of statements directly to justifications, however it differs from the sequential flow that narrative writers adopt. On the other hand, some participants found the flow-chart format to be a complex format to write, in spite of it being visually interesting.

In relation to the second research question, the cognitive factors that determine participants' preferences of proof format type appear to be within their sensory register's component most notably the learners' learning styles. Based on perceptions during the study, verbal/linguistic learners like S1 enjoyed the paragraph rationale format due to the detailed structure it followed, composed of narrations. The flow-chart format bore little space for interpretation for visual/spatial learners (S4, S5, S6) since visual representation depicted the logical relationships. Such variability in purpose points to the necessity of accounting for diversity of individual cognitive abilities in regard to mathematics education.
Addressing the third research question PMTs do appreciate some formats for self-understanding for example flow-chart or personal interest it has been found that they favour the two columns for teaching because it is simple, efficient and effective communication. This indicates the willingness of PMTs to exercise flexibility in their teaching methods so as to accommodate effective delivery and student understanding. This flexibility is important for prospective teachers which means PMTs become aware that there is other focus in mathematics teaching other than just the content itself which always requires some alterations in pedagogy to be effective and efficient.

In mathematics education, the conclusion suggests the use of pedagogical pluralism as a form of activity. There are proofs that learners have diverse cognitive preferences therefore proof in mathematics should be taught in varied ways. It was noted that the use of paragraph, two-column or flow chart writing formats in the teaching of proof has great meaning in the understanding of the work by the students. This means that the teaching and the learning process can now be more holistic and wider.

But the study is too limited in scope as well. It may also be that the results of PMTs' experiences might not be extrapolated to all educational contexts and populations owing to a small sample size and certain contextual characteristics. This would also mean including in future work multisecular studies exploring the downside influence of these preferences on the PMTs instructional practices and the way these preferences affect the academic performance of students in different educational establishments.

It will also be useful for further investigation to evaluate embedding such a multi-format approach in bigger classrooms, while at the same time identifying the challenges that this may pose. Also, with the increased adoption of online education, future research could look into how such motivations would work in distance learning and how to better adjust online systems to fit various proof formats.

To conclude, this study has provided empirical evidence that there is a case for a more dynamic approach with multiple representations in the teaching of mathematical proofs. It follows that by taking into consideration individual learning preferences, it is more likely that deep mathematical understanding will be developed and the effectiveness of instruction will be maximised, thus allowing to educate more flexible and more mathematically competent people.

Acknowledgment:
This research was supported by internal research funding from Universitas Negeri Malang with contract No. 5.4.545/UN32.20.1/LT/2023.

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