

Solutions of the equation $x^2 - (p^2q^2 \pm ap)y^2 = k^t$

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Abstract: In this note, the Diophantine equation $x^2 - (p^2q^2 \pm ap)y^2 = k^t$ has been solved, and its positive integer solutions have been stated in terms of generalized Fibonacci, generalized Lucas, generalized Pell, and generalized Pell-Lucas sequences. We have discovered units for $Z[D]$ in terms of the above sequences of numbers.

Keywords: Diophantine equations, Generalized Fibonacci numbers, Generalized Lucas numbers, Generalized Pell numbers, Generalized Pell-Lucas numbers, Pell equations.

1. Introduction

Since ancient times, numerous mathematicians have studied number sequences. Particular attention has been paid to recurrent sequences like the Fibonacci, Lucas, Pell, or Pell-Lucas sequences, which are described below:

Fibonacci number sequence

$$F_{n+2} = F_{n+1} + F_n \text{ for } n \geq 0 \text{ with } F_0 = 0 \text{ and } F_1 = 1.$$

Lucas number sequence

$$L_{n+2} = L_{n+1} + L_n \text{ for } n \geq 0 \text{ with } L_0 = 2 \text{ and } L_1 = 1.$$

Pell number sequence

$$\Phi_{n+2} = 2\Phi_{n+1} + \Phi_n \text{ for } n \geq 0 \text{ with } \Phi_0 = 0 \text{ and } \Phi_1 = 1.$$

Pell-Lucas number sequence

$$\Psi_{n+2} = 2\Psi_{n+1} + \Psi_n \text{ for } n \geq 0 \text{ with } \Psi_0 = 2 \text{ and } \Psi_1 = 2.$$

In recent years, these sequences of numbers have significant applications in the fields including statistics, music, coding theory, cryptography and communication systems. [2], [3], [4].

As seen in [11,12,13], number sequences are generalized in several ways. One can see some generalizations of Pell and Pell-Lucas numbers. In this paper we will pay attention to the following generalization:

Let s and t be two non-zero integers satisfying $s^2 + 4t > 0$, The generalized Fibonacci and generalized Lucas sequences are, respectively, defined as:

$$F_{n+2}(s, t) = sF_{n+1}(s, t) + tF_n(s, t) \text{ for } n \geq 0 \text{ with } F_0(s, t) = 0, F_1(s, t) = 1, \text{ and}$$

$$L_{n+2}(s, t) = sL_{n+1}(s, t) + tL_n(s, t) \text{ for } n \geq 0 \text{ with } L_0(s, t) = 2, L_1(s, t) = 1,$$

Binet's formulae for these sequences are:

$F_n(s, t) = \frac{\alpha^n - \beta^n}{\alpha - \beta}$, $L_n(s, t) = \alpha^n + \beta^n$, where α and β are the roots of equation $x^2 - sx - t = 0$. α and β verify $\alpha + \beta = s$, $\alpha - \beta = 2\sqrt{s^2 + 4t}$, $\alpha\beta = -t$.

For further details of these sequences, see [5], [6], [7], [17], [18], [19], [20], [21].

Let, now, k and h be two non-zero integers satisfying $k^2 + h > 0$, The generalized Pell and generalized Pell-Lucas sequences are, respectively, defined as:

$\Phi_{n+2}(k, h) = 2k\Phi_{n+1}(k, h) + h\Phi_n(k, h)$ for $n \geq 0$ with $\Phi_0(k, h) = 0$ and $\Phi_1(k, h) = 1$, and

$\Psi_{n+2}(k, h) = 2k\Psi_{n+1}(k, h) + h\Psi_n(k, h)$ for $n \geq 0$ with $\Psi_0(k, h) = 2$ and $\Psi_1(k, h) = 2k$

Binet's formulae for these sequences are:

$\Phi_n(k, h) = 2k \frac{\alpha^n - \beta^n}{\alpha - \beta}$, $\Psi_n(k, h) = \alpha^n + \beta^n$, where α and β are the roots of equation $x^2 - 2kx - h = 0$. α and β verify $\alpha + \beta = 2k$, $\alpha - \beta = 2\sqrt{k^2 + h}$, $\alpha\beta = -h$.

In the plethora of integer sequences, the Fibonacci and Lucas sequences stand out as the two brightest. They have captivated both amateur and expert mathematicians for ages, and they never cease to enchant us with their beauty, myriad practical uses, and omnipresent propensity to appear in completely unexpected and unrelated contexts. They continue to be a fertile ground for creative amateurs and mathematicians alike.

In literature, Fibonacci and Lucas numbers are used to resolve many Diophantine equations.

The quadratic Diophantine equation of the form $x^2 - Dy^2 = N$, where D is square free, generally known as Pell's equation. In this case, if D is square free, the continued fraction expansion of D is periodic and it is given by $D = a_0; \overline{a_1, a_2, \dots, a_{n-1}, a_n}$, where $a_n = 2a_0$ and n is the length of period. As it is known, we denote by $\frac{p_l}{q_l} = [a_0; a_1, a_2, \dots, a_l]$ the l^{th} convergent of D , for $l \geq 0$. In (5), we have obtained some formulas for the integer solutions of the Pell equation $x^2 - y^2 = \pm k^2$ for all $k \geq 1$.

In [9], Bala and Mishra, considered the solutions of two Diophantine equations $x^2 - (p^2q^2 \pm 3p)y^2 = k^t$ and $x^2 - (p^2q^2 \pm 5p)y^2 = k^t$. They generalized a previous result of Guney [10], who found all positive integer solutions of the equations

$x^2 - (a^2b^2 + 2b)y^2 = N$ when $N \in \{\pm 1, \pm 4\}$ in terms of generalized Fibonacci and Lucas sequences.

In this paper, we generalize the results of Bala and Mishra [9], by solving

$$x^2 \pm (p^2q^2 - ap)y^2 = k^t$$

Then we express its positive integer solutions in the current study. Using generalized Lucas, generalized Pell, and generalized Pell-Lucas sequences.

To do, we need the following theorems (1,2,3,2.1):

Theorem 1,

There is no positive integer solution to the equation $x^2 - Dy^2 = -1$ if the length of the period of \sqrt{D} 's continued fraction expansion is even, while the fundamental solution of equation $x^2 - Dy^2 = 1$ is $\frac{p_{n-1}}{q_{n-1}}$.

Theorem 2,

There are infinitely many solutions to the Pell equation $x^2 - Dy^2 = 1$ if D is a natural number that is not a perfect square. All positive solutions can be obtained by the formula

$$x_n + y_n\sqrt{D} = (x_1 + y_1\sqrt{D})^n, \text{ for all } n > 1,$$

where (x_1, y_1) is fundamental solution of equation $x^2 - Dy^2 = 1$.

Theorem 3,

Let N be a positive integer and (u_1, v_1) be the fundamental solution of equation $x^2 - Dy^2 = N$. Then all positive solutions of $x^2 - Dy^2 = N$ can be obtained by the formula

$x_n + y_n\sqrt{D} = (u_1 + v_1\sqrt{D})(x_1 + y_1\sqrt{D})^n$, for all $n > 1$, where (x_1, y_1) is fundamental solution of equation $x^2 - Dy^2 = 1$.

We are now ready to present our main results.

2. Main Results*2.1. Theorem*

Let $D = p^2q^2 - ap$, with $q > a$ being a multiple of a and p, q being positive integers chosen in such a way that D is not a perfect square, then

1. $\sqrt{D} = \sqrt{p^2q^2 - ap} = [pq - 1; 1, \frac{2(q-a)}{a}, 1, 29pq - 1]$.
2. The fundamental solution of the Pell equation $x^2 - Dy^2 = 1$ is $(x_1, y_1) = (\frac{2pq^2 - a}{a}, \frac{2q}{a})$.
3. All positive integer solution (x_n, y_n) of the Pell equation $x^2 - Dy^2 = 1$ are provided by

$$x_n = \frac{\alpha^n + \beta^n}{2} = \frac{1}{2} l_n(s, t) = \frac{1}{2} l_n\left(2\frac{2pq^2 - a}{a}, -1\right), \text{ and } y_n = \frac{\alpha^n - \beta^n}{2\sqrt{D}} = \frac{2q}{a} F_n(s, t) = \frac{2q}{a} F_n\left(2\frac{2pq^2 - a}{a}, -1\right) = \frac{q}{2pq^2 - a} \Phi_n\left(2\frac{2pq^2 - a}{a}, -1\right), \text{ for all } n \geq 1.$$

1. The Fundamental solution of Pell equation $x^2 - Dy^2 = k^t$ is

$$(x_1, y_1) = \left(\frac{2pq^2 - a}{a} k^{\frac{t}{2}}, \frac{2q}{a} k^{\frac{t}{2}}\right).$$

2. All positive solutions of Pell equation $x^2 - Dy^2 = k^t$ is given by

$$x_{n+1} = \frac{2pq^2 - a}{a} k^{\frac{t}{2}} x_n + 2q \frac{p^2q^2 - ap}{a} k^{\frac{t}{2}} y_n \text{ and } y_{n+1} = \frac{2q}{a} k^{\frac{t}{2}} x_n + \frac{2pq^2 - a}{a} k^{\frac{t}{2}} y_n \text{ for all } n \geq 1$$

Proof: 1.

$$\sqrt{D} = \sqrt{p^2q^2 - ap}$$

$$= (pq - 1) + \sqrt{p^2q^2 - ap} - (pq - 1)$$

$$\begin{aligned}
&= (pq - 1) + \frac{1}{\frac{\sqrt{p^2q^2-ap}+(pq-1)}{2pq-ap-1}} \\
&= (pq - 1) + \frac{1}{1 + \frac{\sqrt{p^2q^2-ap}-(pq-ap)}{2pq-ap-1}} \\
&= (pq - 1) + \frac{1}{1 + \frac{1}{\frac{\sqrt{p^2q^2-ap}+(pq-ap)}{ap}}} \\
&= (pq - 1) + \frac{1}{1 + \frac{1}{\frac{2(q-a)}{a} + \frac{\sqrt{p^2q^2-ap}-(pq-ap)}{ap}}} \\
&= (pq - 1) + \frac{1}{1 + \frac{1}{\frac{2(q-a)}{a} + \frac{1}{1 + \frac{1}{2(pq-1) + \sqrt{p^2q^2-ap}-(pq-1)}}}}} \\
&= \left[pq - 1; 1, \frac{2(q-a)}{a}, 1, 2(pq-1) \right]
\end{aligned}$$

2. The period length of \sqrt{D} 's continued fraction expansion is 4, making it even. Then, according to Theorem 1.1, the fundamental solution of equation $x^2 - Dy^2 = 1$ is given by $\frac{p_3}{q_3}$.

Let

$$\frac{p_3}{q_3} = \left[pq - 1; 1, \frac{2(q-a)}{a}, 1 \right] = (pq - 1) + \frac{1}{1 + \frac{2(q-a)}{a} + 1} = \frac{2pq^2 - a}{2q}.$$

$$\text{Then, } (x_1, y_1) = \left(\frac{2pq^2 - a}{a}, \frac{2q}{a} \right).$$

3. According to Theorem 1.1, all of the positive integral solutions to the equation $x^2 - Dy^2 = 1$ is provided by

$$\begin{aligned}
x_n + y_n\sqrt{D} &= \left(\frac{2pq^2 - a}{a} + \frac{2q}{a}\sqrt{D} \right)^n, \text{ and} \\
x_n - y_n\sqrt{D} &= \left(\frac{2pq^2 - a}{a} - \frac{2q}{a}\sqrt{D} \right)^n
\end{aligned}$$

Set $\alpha = \frac{2pq^2 - a}{a} + \frac{2q}{a}\sqrt{D}$ and $\beta = \frac{2pq^2 - a}{a} - \frac{2q}{a}\sqrt{D}$, then $\alpha + \beta = 2\left(\frac{2pq^2 - a}{a}\right)$ and $t = -\alpha\beta = -1$. Thus, for all $n \geq 1$.

$$x_n = \frac{\alpha^n + \beta^n}{2} = \frac{1}{2}L_n(s, t) = \frac{1}{2}L_n\left(2\frac{2pq^2 - a}{a}, -1\right) = \frac{1}{2}\Psi_n\left(\frac{2pq^2 - a}{a}, -1\right),$$

and

$$y_n = \frac{\alpha^n - \beta^n}{2\sqrt{D}} = \frac{2q}{a} F_n(s, t) = \frac{2q}{a} F_n\left(2 \frac{2pq^2 - a}{a}, -1\right) = \frac{q}{2pq^2 - a} \Psi_n\left(\frac{2pq^2 - a}{a}, -1\right),$$

For all $n \geq 1$.

4. It is easy to verify (4) from (1).
5. The fifth part can be proved using the theorem.

2.2. Corollary

Let $D = p^2q^2 - ap$, with $q > a$ being a multiple of a and p, q being positive integers chosen in such a way that D is not a perfect square, then, the equation $x^2 - Dy^2 = 1$ has no positive integer solution.

Proof:

It is simple to conclude using the theorem.;

2.3. Theorem

1. Let $D = p^2q^2 + ap$, with $q > a$ being a multiple of a and p, q being positive integers chosen in such a way that D is not a perfect square, then, $\sqrt{D} = \sqrt{p^2q^2 + ap} = \left[pq; \frac{2q}{a}, 2pq\right]$.

2. The fundamental solution of the Pell equation $x^2 - Dy^2 = 1$ is

$$(x_1, y_1) = \left(\frac{2pq^2 + a}{a}, \frac{2q}{a}\right).$$

3. All positive integer solution (x_n, y_n) of the Pell equation $x^2 - Dy^2 = 1$ are provided by

$$x_n = \frac{\alpha^n + \beta^n}{2} = \frac{1}{2} L_n(s, t) = \frac{1}{2} L_n\left(2 \frac{2pq^2 - a}{a}, -1\right) = \frac{1}{2} \Psi_n\left(\frac{2pq^2 - a}{a}, -1\right),$$

and

$$y_n = \frac{\alpha^n - \beta^n}{2\sqrt{D}} = \frac{2q}{a} F_n(s, t) = \frac{2q}{a} F_n\left(2 \frac{2pq^2 + a}{a}, -1\right) = \frac{q}{2pq^2 - a} \Phi_n\left(2 \frac{2pq^2 + a}{a}, -1\right),$$

for all $n \geq 1$.

4. The Fundamental solution of Pell equation $x^2 - Dy^2 = k^t$ is

$$(x_1, y_1) = \left(\frac{2pq^2 + a}{a} k^{\frac{t}{2}}, \frac{2q}{a} k^{\frac{t}{2}}\right).$$

5. All positive solutions of Pell equation $x^2 - Dy^2 = k^t$ are given by

$$x_{n+1} = \frac{2pq^2 + a}{a} k^{\frac{t}{2}} x_n + 2q \frac{p^2q^2 + ap}{a} k^{\frac{t}{2}} y_n$$

and

$$y_{n+1} = \frac{2q}{a} k^{\frac{t}{2}} x_n + \frac{2pq^2 + a}{a} k^{\frac{t}{2}} y_n$$

for all $n \geq 1$.

Proof:

The proof is similar to theorem 1.

3. Applications

One application of our results is to find the units of $\mathbb{Z}[\sqrt{D}]$. The unity, 1 , and units are those invertible elements. In the following, we will find units of $\mathbb{Z}[\sqrt{D}]$, where $D = p^2q^2 \pm ap$, with $q > a$ being a multiple of a and p, q being positive integers chosen in such a way that D is not a perfect square.

Let $u + v\sqrt{D}$, where $u, v \in \mathbb{Z}$ be a unit element in $\mathbb{Z}[\sqrt{D}]$, then there exists some $c + d\sqrt{D} \in \mathbb{Z}[\sqrt{D}]$ such that

$$(u + v\sqrt{D})(c + d\sqrt{D}) = 1.$$

So, $c + d\sqrt{D} = \frac{1}{u+v\sqrt{D}} = \frac{u-v\sqrt{D}}{u^2-v^2D}$, then if we consider the quantity $N(u + v\sqrt{D}) = u^2 - v^2D$, one can verify that N is multiplicative, and then if $u + v\sqrt{D}$ is invertible, N must be invertible too, so it is either 1 or -1 . Then, we get $u^2 - Dv^2 = \pm 1$.

Further $u^2 - Dv^2 = -1$ has no positive solution. To solve $u^2 - Dv^2 = 1$, we apply our theorems.

Let for example, $p = 2, q = 14, a = 7$, then $D = 2^2(14)^2 - 7 \cdot 14 = 686$ which is, clearly, not a perfect square.

Fundamental solution of $x^2 - 868y^2 = 1$ is $\left(\frac{2 \cdot 2(14)^2 - 7}{7}, \frac{2 \cdot 14}{7}\right) = (195, 4)$.

All positive integer solutions of $x^2 - 868y^2 = 1$ are

$$\begin{cases} x_n = \frac{1}{2}L_n(390, -1) = \frac{1}{2}\Psi_n(195, -1) \\ y_n = 4F_n(390, -1) = \frac{1}{195}\Phi_n(195, -1). \end{cases}$$

We will then be required to find F_n and L_n for all $n > 1$. Let

$$\begin{cases} L_n(390, -1) = 390L_{n-1}(390, -1) - L_{n-2}(390, -1), \\ \forall n > 2, L_0(390, -1) = 2, L_1(390, -1) = 390 \\ F_n(390, -1) = 390F_{n-1}(390, -1) - F_{n-2}(390, -1) \\ \forall n > 2, F_0(390, -1) = 0, F_1(390, -1) = 1 \end{cases}$$

So, $195 + 4\sqrt{686}$ is a unit of $\mathbb{Z}[\sqrt{686}]$. As $L_2(390, -1) = 152098$ and $F_2(390, -1) = 390$, then $76049 + 1560\sqrt{686}$ is another unit of $\mathbb{Z}[\sqrt{686}]$, and so on. We can clearly express units of $\mathbb{Z}[\sqrt{686}]$ in terms of generalized Fibonacci, generalized Lucas, generalized Pell, and generalized Pell-Lucas numbers.

4. Conclusion

The Diophantine equation $x^2 - (p^2q^2 \pm ap)y^2 = k^t$ has been solved, and its positive integer solutions have been stated in terms of generalized Fibonacci, generalized Lucas, generalized Pell, and generalized Pell-Lucas sequences. We have discovered units for $\mathbb{Z}[\sqrt{D}]$ in terms of the above sequences of numbers.

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