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Solving fuzzy fractional programming problems by VNS algorithm using modified Kerre's inequality

Dhurgam Kalel Ibrahim Alsaad[1], Aliasghar Foroughi[1*], Khatere Ghorbani-Moghadam[2], Reza Ghanbari[3]
[1]Department of Mathematics, University of Qom, Qom, Iran; aalimath2018@gmail.com (D.K.I.A.) a-foroughi@qom.ac.ir (A.F.).
[2]Mosaheb Institute of Mathematics, Kharazmi University, Tehran, Iran; k.ghorbani@khu.ac.ir (K.G.M.).
[3]Faculty of Mathematical Sciences, Department of Applied Mathematics, Ferdowsi University of Mashhad, Mashhad, Iran; rghanbari@um.ac.ir (R.Z.).

Abstract: Here, we propose a new variable neighbourhood search (VNS) algorithm for solving fractional fuzzy number linear programming problems (FFNLPPs). We make use of modified Kerre's inequality for comparison of LR fuzzy numbers. In our proposed algorithm, we introduced a new local search defined based on descent directions, which are found by solving four crisp mathematical programming problems. In several methods, a fuzzy fractional optimization problem is converted to a crisp problem. Still, in our proposed method, using modified Kerre's inequality, the fuzzy optimization problem is solved directly, without changing it to a crisp program. To show the effectiveness of our method, we compare our proposed algorithm with other available methods.
Keywords: *Fractional linear optimization, Modified Kerre's inequality, VNS algorithm.*

1. Introduction

Charnes and Cooper [4] first formulated a linear fractional programming problem as an optimization problem of a ratio of linear functions subject to linear constraints. Such linear fractional objectives (i.e., ratio objectives that have a linear numerator and denominator) are helpful in production planning, financial and corporate planning, health care, hospital planning, and so forth. Examples of fractional objectives in production planning include inventory/sales, output/employee, etc. For single objective, linear fractional programming, the Charnes and Cooper [4] transformation can transform the problem into a linear programming problem. In many real-world situations, some parameters of a linear program are given by experts. However, experts and decision-makers frequently are not aware of the precise values of the parameters.

Since some optimization problems contain parameters with imprecise values (see [3]), fuzzy number linear programs (FNLPs) are handy tools for modelling and solving real-world problems.

Arya et al. [1] presented a fuzzy based branch-bound approach is attempted for solving multi-objective linear fractional (MOLF) optimization problems. The original MOLF optimization problem is converted into an equivalent fuzzy multi-objective linear fractional (FMOLF) optimization problem. Then, branch and bound techniques were applied to the FMOLF optimization problem. The feasible space of FMOLF optimization problem was bounded by triangular simplex space. The weak duality theorem was used to generate the bound for each partition and feasibility conditions were applied to neglect one partition in each step. After finite number of steps, a fuzzy efficient (Pareto-optimal) solution was obtained for FMOLF optimization problem which was also efficient (Pareto-optimal) solution of the original MOLF optimization problem. Some theoretical validations were also established for the proposed approach on FMOLF optimization problem. Chinnadurai and Muthukumar [5] proposed using the (α,r) acceptable optimal value for a linear fractional programming problem with fuzzy coefficients and fuzzy decision variables, as well as developing a method for computing them. To obtain acceptable (α,r) optimal values, they take an α-cut on the objective function and r-cut on the

* Correspondence: a-foroughi@qom.ac.ir

constraints. They then formulated an equivalent bi-objective linear fractional programming problem to calculate the upper and lower bounds of the fully fuzzy linear fractional programming (LFP) problem. Using the upper and lower bounds obtained, they constructed the membership functions of the optimal values numerically. Wang et al. [33] obtained the solution to bi-level linear fractional programming problem (BLFP) using an optimization algorithm based on the duality gap of the lower level problem. In their algorithm, the bi-level linear fractional programming problem was transformed into an equivalent single-level programming problem by forcing the dual gap of the lower level problem to zero. Then, by obtaining all vertices of a polyhedron, the single-level programming problem could be converted into a series of linear fractional programming problems. Pal et al. [17], presented a goal programming (GP) procedure for fuzzy multi-objective linear fractional programming (FMOLFP) problems. In the proposed approach, which is motivated by Mohamed (Fuzzy Sets and Systems 89 (1997) 215), GP model for achievement of the highest membership value of each of fuzzy goals defined for the fractional objectives is formulated. In the solution process, the method of variable change on the under- and over- deviational variables of the membership goals associated with the fuzzy goals of the model is introduced to solve the problem efficiently using linear goal programming (LGP) methodology. Yang et al. [34], presented to solve fuzzy multi-objective linear fractional programming (FMOLFP) problems through an approach based on superiority and inferiority measures method (SIMM). In the model for the proposed approach, each of fuzzy goals defined for the fractional objectives and some constraints have fuzzy numbers. To achieve the highest membership value, SIMM was adopted to deal with fuzzy number in constraints, then a linear goal programming methodology was introduced to solve the problem in which the fractional objectives was fuzzy goals. A case of agricultural planting structures optimization problem was solved to illustrate the application of the algorithm. Sakawa and Kato [24], by considering the experts' vague or fuzzy understanding of the nature of the parameters in the problem formulation process, multi-objective linear fractional programming problems with block angular structure involving fuzzy numbers are formulated. Sakawa and Kato [24] proposed the following model:

$$\min\ z_1(x,\tilde{c}_1,\tilde{d}_1)\ldots\min\ z_k(x,\tilde{c}_k,\tilde{d}_k) s.t. \quad (1)$$
$$\tilde{A}_1x_1+\cdots+\tilde{A}_px_p \le \tilde{b}_0, \tilde{B}_1x_1 \le \tilde{b}_1, . \quad . \quad \tilde{B}_px_p \le \tilde{b}_p, x_j \ge 0, j=0,\cdots,p.$$

Where, x_j, $j=1,\cdots,p$, are n_j dimensional column vectors of decision variables, $\tilde{A}_1x_1+\cdots+\tilde{A}_px_p \le \tilde{b}_0$ are m_0-dimensional coupling constraints, and $\tilde{A}_j$, $j=1,\cdots,p$ are $m_0 \times n$ fuzzy coefficient matrices. $\tilde{B}_jx_j \le \tilde{b}_j$ are m_j-dimensional block constraints with respect to x_j, $\tilde{B}_j$, $j=1,\cdots,p$ are $m_j \times n_j$ fuzzy coefficient matrices and $\tilde{b}_j$, $j=1,\cdots,p$, are m_j-dimensional column fuzzy vectors. Furthermore, $z_1(x,\tilde{c}_1,\tilde{d}_1),\cdots,z_k(x,\tilde{c}_k,\tilde{d}_k)$ are k distinct fuzzy linear fractional objective functions defined by :

$$z_i(x,\tilde{c}_i,\tilde{d}_i)=\frac{p_i(x,\tilde{c}_i)}{q_i(x,\tilde{d}_i)}=\frac{\tilde{c}_{i1}x_1+\cdots+\tilde{c}_{ip}x_p+\tilde{c}_{i,p+1}}{\tilde{d}_{i1}x_1+\cdots+\tilde{d}_{ip}x_p+\tilde{d}_{i,p+1}}.$$

Using the a-level sets of fuzzy numbers, the corresponding non-fuzzy α-multi-objective linear fractional programming problem is introduced. The fuzzy goals of the decision maker for the objective functions are quantified by eliciting the corresponding membership functions, including nonlinear ones. Through the introduction of extended pareto optimality concepts, if the decision maker specifies the degree α and the reference membership values, the corresponding extended pareto optimal solution can be obtained by solving the minimax problems for which the Dantzig-Wolfe decomposition method and Ritter's partitioning procedure are applicable. Jiao et al. [14], presented a branch-and-bound algorithm for globally solving a wide class of generalized linear fractional programming problems (GLFP). This class included such problems as: minimizing a sum, or error for product of a finite number of ratios of linear functions, linear multiplicative programming, polynomial programming, etc. over nonconvex feasible region. First a problem (Q) was derived equivalent to problem (GLFP). In the algorithm, lower bounds were derived by solving a sequence of linear relaxation programming problems, which was based on the construction of the linear lower bounding functions for the objective function and constraint functions of the problem (Q) over the feasible region. Shen et al. [26], proposed an interactive fuzzy programming method for seeking a satisfactory solution for multi-objective two-level linear fractional programming problems in which the decision makers in the upper and lower levels have several objectives, by first setting up the fuzzy goals for several objectives of each decision maker and seeking a satisfactory solution for the degree of satisfaction of two decision makers in a cooperative manner. In the proposed method, the decision maker at the upper level sets the minimal satisfactory levels for each fuzzy goal. The decision-maker at the lower level determines the aspiration levels. The minimal satisfactory levels were treated as a constraint and the solution closest to the aspiration levels of the decision maker at the lower level is computed. The ratio of the aggregative degrees of satisfaction of the decision makers in the upper and lower levels with the obtained solution is evaluated by using partial information on the preference of the decision makers. Chun-Feng and Pei-Ping [6], presented an efficient branch and bound method for general linear fractional problem (GFP). First, using a transformation technique, an equivalent problem (EP) of GFP was derived, then by exploiting structure of EP, a linear relaxation programming (LRP) of EP was obtained. To implement the algorithm, primary computation involve solving a sequence of linear programming problem, which could be solved efficiently. The proposed algorithm converged to the global maximum through the successive refinement of the solutions of a series of linear programming problems. D'Amato and Bernstein [8], used retrospective cost optimization to determine linear fractional transformations (LFTs). This method used an adaptive controller in feedback with a known system model. The goal was to identify the feedback portion of the LFT by adapting the controller with a retrospective cost. D'Amato and Bernstein [8] demonstrated this method on numerical examples of increasing complexity, ranging from linear examples with unknown feedback terms to nonlinear examples. Sheikhi and Ebadi [27] presented a novel method for solving fractional transportation problems (FTPs) with fuzzy numbers using a ranking function. The proposed method introduces a transformation technique that converts an FTP with fuzzy numbers into an FTP with crisp numbers by employing the robust ranking technique. Then, they formulated two transportation problems, one for maximization and another for depreciation, based on the given FTP. See other works in fuzzy in [2, 11, 12, 19, 20, 22, 21, 25].

A type of fractional fuzzy number linear programming problem (FFNLPP) can be described as follows:

$$\max \ \frac{\tilde{a}x+\tilde{b}}{\tilde{c}x+\tilde{d}} s.t. \qquad (2)$$
$$Ax \leq h, x \geq 0,$$

where, $\tilde{a} \in F(\mathbb{R}^n)$, $\tilde{c} \in F(\mathbb{R}^n)$, $\tilde{b} \in F(\mathbb{R})$, $\tilde{d} \in F(\mathbb{R})$, $x \in \mathbb{R}^n$, $A \in \mathbb{R}^{m\times n}$ and $h \in \mathbb{R}^m$, with $F(\mathbb{R})$ being the set of all fuzzy numbers.

For solving (2), we propose a new fuzzy VNS algorithm with its local search intended to find a feasible solution by using increasing directions, leading to the value of the objective function being more significant than the current solution. In our proposed algorithm, we compare the fuzzy value of the objective function using the modified Kerre's inequality. Using Kerre's inequality, we can solve the problem (2) directly without changing it to the crisp problem [10].

The rest of our research is structured in the following manner. In Section 2, we provide some necessary definitions, properties of fuzzy ordering and fundamental aspects of modified Kerre's method. We present our model in Section 3. In Section 4, we first talk about our VNS algorithm for solving our proposed model, then to make the algorithm more clear we implement our proposed algorithm on an example. We test our algorithm on several examples in Section 5, and conclude in Section 6.

2. Preliminaries

Here, we provide an overview of the basic principles of fuzzy set theory.

2.1. Definitions and Notation

Definition 1 *[35] Let $\mathbb{R}$ be a collection of objects denoted by x. Then $\tilde{A}$ is called a fuzzy set in $\mathbb{R}$, if $\tilde{A}$ is a set of ordered pairs,*

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$$\tilde{A} = \{(x, \mu_{\tilde{A}}(x)) \mid x \in \mathbb{R}\},$$

where $\mu_{\tilde{A}}(x) \in [0,1]$ is called the membership function or grade of membership of x in $\tilde{A}$.

Definition 2 *[29] A fuzzy number is a fuzzy quantity $\tilde{A}$ satisfying the following conditions :*

.1 $\mu_{\tilde{A}}(x) = 1$, for exactly one x .

.2 The support $\{x: \mu_{\tilde{A}}(x) > 0\}$ of $\tilde{A}$ is bounded .

.3 The α-cuts of $\tilde{A}$ are closed intervals .

Definition 3 *[35] A fuzzy number $\tilde{A}$ is of LR-type if there exist shape functions L (for left), R (for right), and scalars $\alpha > 0$ and $\beta > 0$ such that*

$$\mu_{\tilde{A}}(x) = \begin{cases} L\left(\frac{a-x}{\alpha}\right), & x \le a, \\ R\left(\frac{x-a}{\beta}\right), & x \ge a. \end{cases}$$

The mean value of $\tilde{A}$, a, is a real number, and α and β are called the left and right spreads, respectively. Here, $\tilde{A}$ is denoted by $(a-\alpha/a/a+\beta)_{LR}$.

Remark 1 *Based on Definition 3, another representation of an LR fuzzy number $\tilde{A}$ is $\tilde{A} = (A_L, A_R)$, where A_L is a shape function for the left arm and A_R is a shape function for the right arm .*

Definition 4 *[3] A triangular fuzzy number $\tilde{A}$ is defined by three real numbers $a < b < c$, where the base of the triangle is the interval $[a, c]$ and its vertex is at $x = b$.*

Remark 2 *Another representation of a triangular fuzzy number $\widetilde{M}$ is $\widetilde{M} = (M_L, M_R)$, where M_L and M_R are the functions for the left arm and the right arm of triangular fuzzy number $\widetilde{M} = (a/b/c)$, respectively .*

Note 1 *In the rest of our work, triangular fuzzy numbers will be represented as $\tilde{A} = (a/b/c)$. We also denote a real number a by $\tilde{A} = (a/a/a)$.*

Theorem 1 *[35] Let $\widetilde{M} = (m^L/m/m^R)_{LR}$, let $\widetilde{N} = (n^L/n/n^R)_{LR}$, and let $\lambda \in \mathbb{R}^+$. Then ,*

.1 $\lambda\widetilde{M} = (\lambda m^L/\lambda m/\lambda m^R)_{LR}$.

.2 $-\widetilde{M} = (-m^R/-m/-m^L)_{LR}$.

.3 $\widetilde{M} \oplus \widetilde{N} = (m^L + n^L/m + n/m^R + n^R)_{LR}$.

2.1.1. Modified Kerre's Inequality

Kerre's inequality is recognized as a highly efficient method for assessing fuzzy number comparisons [32]. Ghanbari et al. [10] introduced a modified version of Kerre's inequality and established straightforward formulas for comparing fuzzy triangular numbers, as stated in the following theorem.

Theorem 2 *[10] Consider $\widetilde{M} = (a/b/c)$ and $\widetilde{N} = (a'/b'/c')$ as two triangular fuzzy numbers with $b \le b'$. Then the following assertions hold:*

.1 If $c \le a'$, then

$$r(\widetilde{M}, \widetilde{N}) = \frac{c'-a'}{2} + \frac{c-a}{2}. \tag{3}$$

.2 If $b = b'$, then

$$r(\widetilde{M}, \widetilde{N}) = \frac{c'+a'}{2} - \frac{c+a}{2}. \tag{4}$$

.3 If $b < b'$ and $a' < c$ then

$$r(\widetilde{M},\widetilde{N}) = \frac{c'-a'}{2} + \frac{c-a}{2} - \bar{y}(c-a'), \tag{5}$$

where $\bar{y} = M_R(\bar{x}) = N_L(\bar{x})$ in which $\bar{x}$ is the length of the intersection point of M_R and N_L and defined as follows:

$$\bar{x} = \frac{b'c-ba'}{(b'-a')+(c-b)}.$$

So, $\bar{y}$ in (5) is defined as follows:

$$\bar{y} = \frac{(c-a')}{(b'-a')+(c-b)}. \tag{6}$$

Note 2 *If* $r(\widetilde{M},\widetilde{N}) \geq 0$ *then* $\widetilde{M} \leq \widetilde{N}$, *otherwise* $\widetilde{M} \geq \widetilde{N}$ *.[10]*

Note 3 *Throughout our work,* $<, >$, *and* $=$ *on fuzzy numbers are defined based on our modified Kerre's inequality, and so we show them by* $<^K, >^K$, *and* $=^K$, *respectively [10].*

3. The Proposed Model

We consider the following fractional programming problem:

$$\max \frac{\tilde{a}^T x + \tilde{\alpha}}{c^T x + \beta} (FFNLPP) \quad s.t. \tag{7}$$
$$Ax \leq b, x \geq 0,$$

where, $\tilde{a} \in F(\mathbb{R}^n), \tilde{\alpha} \in F(\mathbb{R})$, $c \in \mathbb{R}^n, \beta \in \mathbb{R}$, $\tilde{a}$ and $\tilde{\alpha}$ are LR fuzzy numbers and $c^T x + \beta > 0$ for any feasible solution x. Let $z = \frac{1}{c^T x+\beta}$ and $y = z^T x$. By getting the ideas from [4], the problem (7) can be converted to the following problem:

$$\max \tilde{a}^T y + \tilde{\alpha} z, s.t. \tag{8}$$
$$Ay - bz \leq 0, c^T y + \beta z = 1, y, z \geq 0,$$

We can rewrite the problem (8) to the vector form as follows:

$$\max [\tilde{a}, \ \tilde{\alpha}] \begin{bmatrix} y \\ z \end{bmatrix}, s.t., \tag{9}$$
$$[A, \ -b^T] \begin{bmatrix} y \\ z \end{bmatrix} \leq 0, [c^T, \ \beta] \begin{bmatrix} y \\ z \end{bmatrix} = 1 [y, z] \geq 0,$$

If we suppose $X = \begin{bmatrix} y \\ z \end{bmatrix}$, $\widetilde{D} = [\tilde{a}, \tilde{\alpha}]$, $B = [A, \ -b^T]$ and $H = [c^T, \beta]$ then we have,

$$\max \tilde{f}(X) = \widetilde{D}X, s.t. \tag{10}$$
$$BX \leq 0, HX = 1 X \geq 0.$$

By solving the problem (10), we can find the so lution of the problem (7).

Example 1 *Consider the following programming problem:*

$$\max \frac{(-43/-3/0)x_1 + (-76/-48/45)x_2 + (-22/15/41)}{14x_1 + 55x_2 + 9}, s.t., \tag{11}$$
$$100x_1 + 59x_2 \leq 359, 24x_1 + 41x_2 \leq 229, x_1, x_2 \geq 0.$$

Let, $z = \frac{1}{14x_1+55x_2+9}$ and $y = zx$. So the problem (11) convert to the following problem:

$$\max (-43/-3/0)y_1 + (-76/-48/45)y_2 + (-22/15/41)z, s.t., \tag{12}$$
$$100y_1 + 59y_2 - 359z \leq 0, 24y_1 + 41y_2 - 229z \leq 0, 14y_1 + 55y_2 + 9z = 1, y_1, y_2, z \geq 0.$$

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Finally, we can rewrite the problem (12) to the following problem:

$$\max \ ((-43,-3,0),(-76,-48,45),(-22,15,41))\begin{pmatrix} y_1 \\ y_2 \\ z \end{pmatrix}, s.t., \tag{13}$$
$$[14,55,9]\begin{pmatrix} y_1 \\ y_2 \\ z \end{pmatrix} = 1 \begin{pmatrix} 100 & 59 & -395 \\ 24 & 41 & -229 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \\ z \end{pmatrix} \leq 0, y_1, y_2, z \geq 0.$$

In the next section we propose VNS algorithm for solving the problem (10).

4. Vns Algorithm for Solving Fuzzy Fractional Programming Problem with Fuzzy Objective Parameter

The VNS algorithm was initially introduced by Mladenovic and Hansen [18]. It is a local search approach that systematically explores the search space by introducing deliberate modifications to neighbourhood structures. Considering the demonstrated effectiveness of VNS and its extensions in addressing combinatorial and continuous optimization problems [18], we propose the utilization of VNS. In each iteration, a specific neighborhood structure, denoted as k, is chosen following the sequence outlined in [10].

A random neighbor s^' is generated in this neighborhood. Afterwards, a descent method is applied to s^'. If the best solution found by descent method, s^'', is better than the best known solution, it is updated and the neighborhood structure is set to the first one. Otherwise, the search continues in the following neighborhood structure. When we explore the last neighborhood structure kmax, the search return to the first neighborhood. This process continues until a stop condition is reached. Some successful examples of application of VNS can also be found in [7, 30, 31].

Based on the definition of the objective function in the $(FFNLPP)$, the value of the objective function for each feasible solution is a fuzzy number. Thus, to compare the solutions in the proposed algorithm (e.g., Step 1-2-3), we use modified Kerre's inequality in Theorem 2. Using our modified Kerre's inequality, the fuzzy optimization problem is solved directly without changing it to a crisp problem.

Algorithm 1 FVNS for solving $\boldsymbol{FFNLPP}$

Inputs: Neighboring structures $\boldsymbol{Nt}$ $(\boldsymbol{t = 1, 2, \cdots, t_{\max}})$, an initial feasible solution $\boldsymbol{x_0}$ and $\boldsymbol{maxiter}$ (maximum number of iterations).

[**output:**] Return $\boldsymbol{x^*}$. [1.]

While $\boldsymbol{k < maxiter}$ do :

.1 $\boldsymbol{t = 1}$ **.**

.2 **If $\boldsymbol{t < t_{\max}}$ then :**

(a) Select one point in the neighborhood of $\boldsymbol{x_0}$ and name it to be $\boldsymbol{x_1}$ $(\boldsymbol{x_1 \in N_t(x_0)})$.

(b) Apply local search on $\boldsymbol{x_1}$ and name the new point as $\boldsymbol{x_2}$.

(c) if $\boldsymbol{\tilde{f}(x_2) \leq^K \tilde{f}(x_0)}$ then let $\boldsymbol{t = t + 1}$ and go to 1-2, **else** go to 1-3 .

.3 $\boldsymbol{x_0 = x_2}$ **.**

.4 $\boldsymbol{k = k + 1}$ **.**

Details of the steps involved in algorithm 1 are described below .

.1 *Neighboring structure* [10]: We define neighboring structure as follows :

$$||x' - x_0|| < t\epsilon, \tag{14}$$

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for $t = 1, \cdots, t_{\max}$, where x' is in the t-th neighborhood of x_0 and ϵ is an arbitrary parameter .

.2 *Selection of a point in the neighborhood of* x_0: Let x' be a point in the neighborhood of x_0. Then, x' is to satisfy in the following conditions :

(a) $(t-1)\epsilon < ||x' - x_0|| \leq t\epsilon$.

(b) $Bx' \leq 0$.

(c) $Hx' = 1$.

(d) $x' \geq 0$.

see details in .[10]

.3 *Local search*[10]: Let x_k represent a feasible solution during iteration k, we want to find a feasible solution with larger value of the objective function with respect to x_k. In other words, we must find s_k such that $\widetilde{D}^T(x_k + s_k) \geq^k \widetilde{D}^T x_k$, where,

$$\begin{cases} \widetilde{D}^T(x_k + s_k) = \\ ({D^L}^T(x_k + s_k)/D^T(x_k + s_k)/{D^R}^T(x_k + s_k)), \\ \widetilde{D}^T x_k = ({D^L}^T x_k / D^T x_k / {D^R}^T x_k). \end{cases}$$

Since the given results for the modified Kerre's method depend on the mean value of the fuzzy numbers, to find s_k we consider the following three cases:

Case 1: $D^T x_k < D^T(x_k + s_k)$,
Case 2: $D^T(x_k + s_k) < D^T x_k$,
Case 3: $D^T(x_k + s_k) = D^T x_k$.

(a) Let $D^T x_k < D^T(x_k + s_k)$, based on Note 2, we must have $r(\widetilde{D}^T x_k, \widetilde{D}^T(x_k + s_k)) \geq 0$, and thus we propose problem (P_1) as follows:

$$\begin{aligned} &\max \ r(\widetilde{D}^T(x_k), \widetilde{D}^T(x_k + s_k)), s.t. \\ &(P_1) \ \ D^T s_k \geq 0, Bs_k \leq -Bx_k, Hs_k = 1 - Hx_k, -s_k \leq x_k. \end{aligned} \tag{15}$$

Note that replacing the constraint $D^T s_k > 0$ with $D^T s_k \geq 0$ has no impact on the solution of (P_1) .

i. Let $(\widetilde{D}^T x_k)_R$ and $(\widetilde{D}^T(x_k + s_k))_L$ have an intersection point. In this case, we have $D^T x_k \leq D^T(x_k + s_k)$ and ${D^L}^T(x_k + s_k) \leq {D^R}^T x_k$, and (15) turns into (P_1') as follows:

$$\begin{aligned} &\max \ (D^R - D^L)^T x_k + \frac{(D^R - D^L)^T}{2} s_k - \frac{((D^R - D^L)^T x_k - {D^L}^T s_k)^2}{(D^R - D^L)^T x_k + (D - D^L)^T s_k}, s.t., \\ &(P_1') \ \ - D^T s_k \leq 0, {D^L}^T s_k \leq (D^R - D^L)^T x_k, Bs_k \leq -Bx_k, Hs_k = 1 - Hx_k, -s_k \leq x_k. \end{aligned} \tag{16}$$

Note that (P_1') is a quadratic fractional programming problem. By using a method that is proposed in [9] (Theorem and Section 3) we solve problem (P_1') .

ii. ${D^R}^T x_k \leq {D^L}^T(x_k + s_k)$, that means two numbers do not have any intersection point (see (3)). By using (3) we have the following program:

$$\begin{aligned} &\max \ (D^R - D^L)^T x_k + \frac{(D^R - D^L)^T}{2} s_k, s.t., \\ &(P_2) \ \ - D^T s_k \leq 0, -{D^L}^T s_k \leq (D^L - D^R)^T x_k, Bs_k \leq -Bx_k, Hs_k = 1 - Hx_k, -s_k \leq x_k. \end{aligned} \tag{17}$$

After removing the constant terms, we have:

$$\begin{aligned} &\max \ \frac{(D^R - D^L)^T}{2} s_k, s.t., \\ &(P_2') \ \ - D^T s_k \leq 0, -{D^L}^T s_k \leq (D^L - D^R)^T x_k, Bs_k \leq -Bx_k, Hs_k = 1 - Hx_k, -s_k \leq x_k. \end{aligned} \tag{18}$$

(b) Let $D^T x_k > D^T(x_k + s_k)$. In this case if $r(\widetilde{D}^T(x_k + s_k), \widetilde{D}^T x_k) \leq 0$ then $\widetilde{D}^T(x_k + s_k) >^k \widetilde{D}^T x_k$. So, we propose the following problem:

$$\min\ r(\widetilde{D}^T(x_k + s_k), \widetilde{D}^T x_k), s.t., \quad (19)$$

$$(P_3)\quad D^T s_k \leq 0, Bs_k \leq -Bx_k, Hs_k = 1 - Hx_k, -s_k \leq x_k.$$

Note that replacing the constraint $D^T s_k < 0$ with $D^T s_k \leq 0$ has no impact on the solution of (P_3) .

i. Let $(\widetilde{D}^T(x_k + s_k))_R$ and $(\widetilde{D}^T x_k)_L$ have an intersection point. In this case, we have $D^T x_k \geq D^T(x_k + s_k)$ and $D^{L^T} x_k \leq D^{R^T}(x_k + s_k)$, the problem (19) turns into (P_3') as follows:

$$\min\ (D^R - D^L)^T x_k + \frac{(D^R - D^L)^T}{2} s_k - \frac{((D^R - D^L)^T x_k + D^{R^T} s_k)^2}{(D^R - D^L)^T x_k + (D^R - D)^T s_k}, s.t., \quad (20)$$

$$(P_3')\quad D^T s_k \leq 0, -D^{R^T} s_k \leq (D^R - D^L)^T x_k B s_k \leq -Bx_k, Hs_k = 1 - Hx_k, -s_k \leq x_k.$$

By using a method that is proposed in [9] (Theorem and Section 3) we solve problem (P_3') .

ii. $D^{R^T}(x_k + s_k) < D^{L^T} x_k$, that means two numbers do not have any intersection point (see (3)). Since $D^{R^T}(x_k + s_k) < D^{L^T} x_k$ and $\widetilde{D}^T(x_k + s_k)$ is located completely on the left side of $\widetilde{D}^T x_k$ and according to (5) $r(\widetilde{D}^T(x_k + s_k), \widetilde{D}^T(x_k)) > 0$, so there is no increasing direction.

(c) Let $D^T x_k = D^T(x_k + s_k)$. So, according to (4), this case is divided into two cases :

i. Suppose $a = D^{L^T} x_k$, $c = D^{R^T} x_k$, $a' = D^{L^T}(x_k + s_k)$ and $c' = D^{R^T}(x_k + s_k)$ in :(4)

$$\max\ \frac{(D^R + D^L)^T}{2} s_k, s.t., \quad (21)$$

$$(P_4)\quad D^T s_k = 0, Bs_k \leq -Bx_k, Hs_k = 1 - Hx_k, -s_k \leq x_k.$$

ii. Suppose $a = D^{L^T}(x_k + s_k)$, $c = D^{L^T}(x_k + s_k)$, $a' = D^{L^T} x_k$, $c' = D^{R^T} x_k$, in :(4)

$$\min\ -\frac{(D^R + D^L)^T}{2} s_k, s.t., \quad (22)$$

$$(P_5)\quad D^T s_k = 0, Bs_k \leq -Bx_k, Hs_k = 1 - Hx_k, -s_k \leq x_k.$$

4. Stopping condition[10]: We stop the proposed FVNS algorithm when reach maxiter successive iterati ons without improvement.

Note 4 *The (21) and (22) problems have the same solution, so you can find the increasing direction by solving either one.*

Finally, to find s_k, we need to solve for P_1^', P_2, P_3^', and P_4. It is clear that P_1^', P_2, P_3^' and P_4 can be solved in parallel. After solving these problems, choose the best s_k. That is, the one given by P_1^', P_2, P_3^' and P_4 that causes the largest increase in the objective function.

Example 2 *Suppose the fuzzy fractional linear programming problem as follows :*

$$\max\ \tilde{f}(x) = \frac{(-97,-73,-31)x_1 + (-37,-27,9)x_2 + (-99,-27,42)x_3 + (-77,-53,14)}{64x_1 + 3x_2 + 12x_3 + 82}, s.t. \quad (23)$$

$$86x_1 + 11x_2 + 86x_3 \leq 280, 73x_1 + 90x_2 + 17x_3 \leq 343, x_1, x_2, x_3 \geq 0.$$

With x_0 as the starting point (which can be generated randomly in the interval [1,5]), using Algorithm 4, after one iteration we obtain:

$$x^0 = [2,2,1], \tilde{f}(x^0) = [-3.6680, -2.3028, 0.0987], x^* = [0, 0.3575, 3.2101], \tilde{f}(x^*) = [-3.3557, -1.2281, 1.2504], r(\tilde{f}(x^0), \tilde{f}(x^*)) = 1.5517. \quad (24)$$

According to Note 2, we can conclude that $\tilde{f}(x^*) >_k \tilde{f}(x^0)$.

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5. Numerical Results

In this section, we demonstrate the effectiveness of the algorithm 4 we have developed. To evaluate its performance, we generated numerous test problems for the FFLPP. These test problems were created using a random generation process in the MATLAB 7.0 programming environment, executed on a notebook equipped with an Intel(R) Core(TM) i5-3210M CPU running at 2.5 GHz, with 4.00 GB of RAM. To introduce fuzzy random coefficients into the objective function, we initiated the process by generating three numbers within the range of [-100,100]. Subsequently, these numbers were sorted in ascending order and employed as the components of the fuzzy number.

Table 1.
Objective function values obtained by various methods on the examples.

dim	Obj(init)	Obj (A) 1]	r(Al1),
10×20	[−12.5611/−6.5882/0.5839]	[−4.7204/−3.8596/−2.6984]	4.081016
15×45	[−13.3151/−4.8021/3.0413]	[−5.7192/−5.0795/−0.4860]	-1.7628
20×50	[−83.3284/−29.3203/19.0412]	[−24.2866/−10.7058/1.1153]	33.5785
25×75	[−37.0272/−18.1645/5.1532]	[−14.7937/−5.1766/−2.6203]	15.0961
60×100	[−117.8211/−42.6375/32.4857]	[−40.4461/−36.0957/−25.3402]	15.7779
25×200	[−76.6690/−72.7284/−29.6852]	[−187.8614/−74.2193/41.3462]	21.5528
40×300	[−56.8188/−21.7506/13.0071]	[−23.6253/−19.7209/−0.8734]	11.5796
300×600	[−222.6613/−88.4877/48.5212]	[−86.3218/−81.8225/−77.2927]	11.6141

In Table 1, the dim column shows the dimensions of the test problems, the column entitled Obj (init) shows the value of the objective function corresponding to the initial solution of Algorithm 1, the column entitled Obj (Al4) shows the values of the objective functions obtained by Algorithm 1 and the column entitled r(Al1,) shows the comparison of the value of objective function of initial solution corresponding to Algorithm 1 with the ones obtained by Algorithm4 using (5). According to the obtained results shown in column labeled as r(Al1,), it is observed (based on Note (2)) that the Algorithm 4 can improve the initial solution.

All problems were also solved by the methods due to [27]. Sheikhi and Ebadi in [27] proposed method introduces a transformation technique that converts an FTP with fuzzy numbers into an FFNLPP with crisp numbers by employing the robust ranking technique. Details o the results are shown in Table 2.

Table 2.
Objective function values obtained by Sheikhi and Ebadi [27] on the examples.

dim	Obj(Al[27])	r(Obj(Al]), Obj(Al[27]))
10×20	[−10.7321/−4.1001/1.2150]	3.2697
15×45	[−14.8912/−6.1221/1.7841]	2.6305
20×50	[−80.1432/−35.5698/16.1201]	5.7479
25×75	[−30.8531/−20.1345/−10.5200]	6.5833
60×100	[−120.2510/−50.3217/25.1432]	12.1784
25×200	[−110.2002/−80.1214/−10.4879]	13.8171
40×300	[−60.1901/−30.3114/20.5417]	5.6262
300×600	[−150.1122/−90.3333/−20.5417]	0.0858

In Table 2, the dim column shows the dimension of problems, the column entitled Obj(Al [27]) shows the value of the objective function corresponding to the solution obtained by Algorithm proposed in [27], and the column entitled r(Al1, Al([27])) shows the comparison of the value of objective function corresponding to Algorithm 4 with the ones obtained by the methods using in [27]. According to the

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obtained results shown in column labeled as r(Al1, Al([27])), it is observed that the objective function values obtained by Algorithm 1 are more significant than the ones due to other methods on all the test problems. The obtained results showed that the objective function values obtained by the proposed algorithm were significant than the ones obtained by the algorithms due to Sheikhi and Ebadi in [27]

6. Conclusions and Future Work

In this paper, we considered a fractional fuzzy linear programming problem, with the assumption that the parameters of the objective function are fuzzy numbers, and we presented a new model with the name FFNLPP. We proposed an FVNS algorithm based on modified Kerre's inequality for solving FFNLPP. We generated and solved some randomly tested examples with triangular fuzzy coefficients, and we concluded that our proposed algorithm could improve the initial solution by increasing the direction we introduced through Kerre's inequality. Also, we compare our proposed algorithm with another method.

Similar to technique that is introduced in [13], by using ABS algorithm the fractional fuzzy linear programming can be simplified and by using of this method we can decrease the number of variables.

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