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Propagation of nonstationary boundary disturbances in a half-plane filled with a homogeneous isotropic elastic medium

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Abstract: This study presents a method for addressing dynamic issues using an elastic half-plane subjected to different sorts of boundary disturbances. The motion of the half-plane is governed by wave equations that pertain to the scalar and non-zero components of the vector elastic displacement potentials. It is assumed that the starting circumstances are zero. An explicit solution to the problem is obtained by using the integral relation between the components of displacements and stresses of the half-plane boundary. This relation is expressed as a two-dimensional convolution with the influence function resulting from the principle of superposition. The properties of the convolution operation in two variables and the theory of generalized functions are used to derive the solution in integral form. Simultaneously, the acquisition of this solution relies on the technique of decomposing the influence function, whereby it is expressed as the multiplication of two components that meet the predetermined essential criteria. Hence, to get conclusive outcomes, it is important to factorize the impact function that has the given characteristics. The process of obtaining the necessary factorization of the influence function relies on describing its image as a multiplication of individual elements. The first derivation of this function was accomplished by the use of the joint inversion technique of the Fourier-Laplace transform, relying on analytical representations of pictures. Consequently, explicit integral formulae were derived to solve the issue and enable the determination of unknown displacements and stresses at any speed range of motion of the boundary conditions interface point. An example of a particular sort of boundary condition is provided to demonstrate the technique for addressing common situations.
Keywords: *Boundary disturbances, Elastic half-plane, Fourier-laplace transform, Homogeneous isotropic elastic medium, Propagation of nonstationary.*

1. Introduction

Continuum dynamics is the most intricate subdivision of mechanics. The study of this subject is relevant since all natural events are known to be non-stationary. The commonly used notions of static and non-stationary processes are only an approximation, typically justified, of actual occurrences. Considering the dynamic characteristics of the environment is often essential, both in terms of quality and quantity. Investigating the phenomenon of wave propagation in confined and partially confined continuous media is a more intricate undertaking compared to researching the same phenomena in unrestricted media. Currently, there is a lack of research on the mechanics of a deformable solid when it comes to two-dimensional non-stationary situations involving the propagation of boundary disturbances in an elastic half-plane. Simultaneously, the advancement of geology, seismology, metal processing technology, and other scientific and technological fields that deal with non-stationary plane problems necessitated the investigation of related mixed problems, where disturbances are defined at the boundary of the medium. Similar issues also occur when examining the phenomena of impact contact between elastic blunt objects and completely rigid or deformable surfaces, as well as in situations involving the effects of moving loads on solid deformable objects, and in cases involving fissures.

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It is important to consider the impact functions associated with concentrated kinematic or force effects while tackling issues of this kind. They are essential solutions of the operators that explain the mathematical model of the item being studied. Generally, the combinations of these elements create a tensor. The tensor's components serve as kernels in integral equations that solve the associated problems. Lamb [1] was the first to address the issue of identifying non-stationary effect functions. The problems associated with finding influence functions are crucial for creating solutions to systems of equations for non-stationary contact issues. Consequently, several academics from both local and international backgrounds have been actively involved in addressing these issues. In 1989, De-sui [2] demonstrated the practical use of Ungar's differential transformation in the field of elastodynamics. Kosloff et al. [3] developed a Fourier-based approach for two-dimensional forward modeling. To address the free surface boundary condition using the Fourier approach, a novel set of wave equations was created. These equations use the stresses as unknowns, rather than the displacements. The solution method included a process of dividing both the spatial and temporal dimensions into discrete segments. The Fast Fourier Transform was used to approximate spatial derivatives, while second-order differencing was used to derive temporal derivatives. The numerical technique was validated by comparing it to the analytic solution for Lamb's issue in two dimensions. Richards [4] discussed fundamental solutions to Lamb's dilemma for a single source point and their significance in three-dimensional investigations of spontaneous fracture growth. Steinfeld and coworkers [5] delivered a comprehensive study on the propagation of 3D waves in an elastic half-space, specifically focusing on the classical approach and the direct boundary element method. Melnikov [6] presented the impact of point sources on perforated compound plates with face convection using influence functions. Churchman and colleagues [7] performed a study on the impact functions of the edge dislocation in a three-quarter plane. Tarlakovskii and Fedotenkov [8] analyzed the movement of a thin elastic spherical Timoshenko shell in three dimensions, caused by nonstationary pressure that is dispersed in an arbitrary manner. A method for dividing the set of equations describing the three-dimensional motion of the shell was suggested. The solution was developed utilizing integral representations with influence functions as kernels. These influence functions were calculated analytically by using series expansions in the eigenfunctions and the Laplace transform. A computational method was developed and used to solve the issue of the effect of time-varying normal pressure on the shell. The findings obtained have practical applications in the building of airplanes, rockets, and several other industrial domains. These applications are particularly relevant for thin-walled shell structural components that are subjected to nonstationary operating circumstances. Tuan et al. [9], [10] studied how non-stationary kinematic disturbances spread from a spherical hollow in the pseudo-elastic Cosserat medium.

The process of solving planar nonstationary problems with moving borders may be simplified by using the superposition principle. This involves studying a two-dimensional boundary integral equation, where the integral operators' kernels are the surface functions that represent the effect of interacting entities. Mikhailova and Fedotenkov [11] introduced a dynamic and symmetrical issue involving the collision of a spherical shell with an elastic half-space. This scenario represents the early phase of their interaction. Shmegera [12] presented the first boundary-value mixed problems for an elastic half-plane, considering the circumstances of contact friction. Suvorov et al. [13] discussed the issue of a stiff body colliding with a half-space that is represented by a Cosserat medium. Tarlakovskii and Fedotenkov [14] performed a study on the dynamic interaction of elastic cylindrical or spherical shells in a two-dimensional setting. Igumnov et al. [15] introduced a problem involving the motion of a surface load across an elastic half-space, which is not stationary. Then, Igumnov and his team [16] presented the use of Boundary-Element Modeling to analyze the behavior of elastic and viscoelastic bodies and media. His research team [17] also devised a boundary element technique for transient anisotropic viscoelastic issues in three dimensions. The Wiener-Hopf method [18] was a very successful analytical technique for solving integral equations of this kind. To find out the mechanical characteristics as well as the mechanical response of structures, there are many different approaches, from analytical methods to approximate methods [19]-[27]. In this work, the solution is achieved by the use of the approach of splitting fundamental solutions [18], which is based on the factorization of the influence function of an elastic half-plane.

The rest of this paper is structured as follows: Section 2 presents a formulation of the proposed problem. Section 3 introduces a solution method. Some examples are introduced in Section 4. Conclusions are summed up in Section 5.

2. Formulation of the Problem

The process of propagation of a plane progressive wave in the direction of the boundary of a half-plane occupied by a homogeneous elastic isotropic medium is considered. The rectangular Cartesian coordinate system *Oxz* is selected so that the *Oz* axis is directed deep into the half-plane, and the *Ox* axis coincides with the boundary of the half-plane $z = 0$.

The equation of motion of the medium in the absence of mass forces has the following form:

$$\rho\frac{\partial^2\mathbf{u}}{\partial t^2}=\left(\lambda+\mu\right)\operatorname{grad}\operatorname{div}\mathbf{u}+\mu\Delta\mathbf{u} \tag{1}$$

To describe the motion of the medium, we use equations in potentials, in which, due to the two-dimensionality of the problem, we should put:

$$\varphi=\varphi(x,z,t),\quad \psi_1=\psi_2\equiv 0,\quad \psi_3=\psi(x,z,t)$$

In this case, we arrive at the following equations for the scalar potential and the non-zero component of the vector potential (the dots indicate derivatives with respect to time):

$$\begin{gathered}\gamma_1^2\ddot{\varphi}=\Delta\varphi,\quad \gamma_2^2\ddot{\psi}=\Delta\psi,\quad x\in R,\quad z>0,\quad t>0;\\ \Delta=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial z^2}\end{gathered} \tag{2}$$

We use dimensionless parameters (they are indicated by a stroke):

$$x'=\frac{x}{L},\quad z=\frac{z}{L},\quad u_i'=\frac{u_i}{L},\quad \tau=\frac{c*t}{L},\quad \sigma_{ij}'=\frac{\sigma_{ij}}{\lambda+2\mu}\quad (i,j=1,2,3),$$

$$\chi=\frac{\lambda}{\lambda+2\mu}=\frac{\nu}{1-\nu},\quad \gamma_m=\frac{c_*}{c_m}\quad (m=1,2),\quad \eta=\frac{c_1}{c_2}=\frac{\gamma_2}{\gamma_1}=\sqrt{\frac{2}{1-\chi}},$$

$$r'=\frac{r}{L},\quad \varphi'=\frac{\varphi}{L^2},\quad \psi'=\frac{\psi}{L^2}$$

Here u_i, σ_{ij} - the components of the displacement vector, stress tensor; c_m - the speeds of propagation of longitudinal and transverse waves in an elastic medium; λ,μ,ν,ρ - parameters Lame, Poisson's ratio, density of the medium; c_* - a parameter having the dimension of velocity; L - characteristic linear dimension.

Omitting the strokes in the designation of dimensionless parameters, we obtain relations that describe the movement of the half-density through potentials, geometric and physical equations of the medium:

$$u_1=\frac{\partial\varphi}{\partial x}-\frac{\partial\psi}{\partial z},\quad u_3=\frac{\partial\varphi}{\partial z}+\frac{\partial\psi}{\partial x},\quad u_2\equiv 0; \tag{3}$$

$$\begin{gathered}\sigma_{11}=\frac{\partial u_1}{\partial x}+\chi\frac{\partial u_3}{\partial z},\quad \eta^2\sigma_{13}=\frac{\partial u_3}{\partial x}+\frac{\partial u_1}{\partial z},\quad \sigma_{33}=\frac{\partial u_3}{\partial z}+\chi\frac{\partial u_1}{\partial x};\\ \sigma_{12}=\sigma_{23}\equiv 0,\quad \sigma_{22}=\frac{\chi}{1+\chi}(\sigma_{11}+\sigma_{33})\end{gathered} \tag{4}$$

At the initial moment of time and at infinity there are no disturbances:

$$\varphi\big|_{\tau=0}=\dot{\varphi}\big|_{\tau=0}=\psi\big|_{\tau=0}=\dot{\psi}\big|_{\tau=0}=0; \tag{5}$$

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$$\varphi = O(1), \quad \psi = O(1), \quad z \to +\infty \tag{6}$$

Let us assume that on the boundary of the half-plane the generalized conditions are satisfied:

$$\begin{aligned} &(\alpha_1 u_1 + \beta_1 \sigma_{13})\big|_{z=0} = q_1(x,\tau); \\ &(\alpha_3 u_3 + \beta_3 \sigma_{33})\big|_{z=0} = q_3(x,\tau), \end{aligned} \tag{7}$$

where $\alpha_i^2 + \beta_i^2 \neq 0$.

All indices here and below take values 1 and 3. If necessary, summation is carried out over them.

The combination of different parameter values in (7) provides all possible boundary conditions. First boundary value problem (kinematic excitation) (when $\alpha_1 = \alpha_3 = 1$ and $\beta_1 = \beta_3 = 0$):

$$u_j\big|_{z=0} = q_j(x,\tau) \tag{8}$$

Second boundary value problem (dynamic excitation) when $\alpha_1 = \alpha_3 = 0$ and $\beta_1 = \beta_3 = 1$:

$$\sigma_{j3}\big|_{z=0} = q_j(x,\tau) \tag{9}$$

Mixed excitation of the first type (when $\beta_1 = \alpha_3 = 0$ and $\alpha_1 = \beta_3 = 1$):

$$u_1\big|_{z=0} = q_1(x,\tau), \quad \sigma_{33}\big|_{z=0} = q_3(x,\tau) \tag{10}$$

Mixed excitation of the first type (when $\alpha_1 = \beta_3 = 0$ and $\beta_1 = \alpha_3 = 1$):

$$\sigma_{13}\big|_{z=0} = q_1(x,\tau), \quad u_3\big|_{z=0} = q_3(x,\tau) \tag{11}$$

3. Solution Method

To solve the problem, we introduce surface influence functions:

$G_{jl}(x,z,\tau) = u_j$ and $\Gamma_{jkl}(x,z,\tau) = \sigma_{jk}$

These surface influence functions are similar to displacement and stress and satisfy the relations (2)-(6), as well as the following conditions on the boundary of the half-plane ($l = 1,3$):

$$\begin{aligned} &(\alpha_1 u_1 + \beta_1 \sigma_{13})\big|_{z=0} = \delta(x)\delta(\tau)\delta_{1l}; \\ &(\alpha_3 u_3 + \beta_3 \sigma_{33})\big|_{z=0} = \delta(x)\delta(\tau)\delta_{3l} \end{aligned} \tag{12}$$

where $\delta(x)$ - the delta function Dirac.

Then displacements and stresses in problem (2)-(7) have integral representations:

$$u_j(x,z,\tau) = G_{jl}(x,z,\tau) ** q_l(x,\tau); \tag{13}$$

$$\sigma_{jk}(x,z,\tau) = \Gamma_{jkl}(x,z,\tau) ** q_l(x,\tau) \tag{14}$$

Note that G_{jl} and Γ_{jkl} are components of tensors of the corresponding second and third ranks.

To determine the influence functions, we apply to the equalities (2)-(7) and (12) taking into account (5) the integral Fourier transforms with respect to coordinate x and Laplace transforms with respect to time τ:

$$\begin{aligned} &\frac{\partial^2 \varphi^{FL}}{\partial z^2} - k_1^2(q^2, s^2)\varphi^{FL} = 0; \\ &\frac{\partial^2 \psi^{FL}}{\partial z^2} - k_2^2(q^2, s^2)\psi^{FL} = 0, \quad z > 0; \\ &k_j(q,s) = \sqrt{q + \gamma_j^2 s}; \end{aligned} \tag{15}$$

$$u_1^{FL}=-iq\varphi^{FL}-\frac{\partial\psi^{FL}}{\partial z},\quad u_3=\frac{\partial\varphi^{FL}}{\partial z}-iq\psi^{FL};\tag{16}$$

$$\sigma_{11}^{FL}=-iqu_1^{FL}+\chi\frac{\partial u_3^{FL}}{\partial z};$$

$$\eta^2\sigma_{13}^{FL}=-iqu_3^{FL}+\frac{\partial u_1^{FL}}{\partial z};\tag{17}$$

$$\sigma_{33}^{FL}=\frac{\partial u_3^{FL}}{\partial z}-iq\chi u_1^{FL};$$

$$\varphi^{FL}=O(1),\quad \psi^{FL}=O(1),\quad z\to+\infty;\tag{18}$$

$$\begin{gathered}(\alpha_1u_1^{FL}+\beta_1\sigma_{13}^{FL})\Big|_{z=0}=\delta_{1l};\\(\alpha_3u_3^{FL}+\beta_3\sigma_{33}^{FL})\Big|_{z=0}=\delta_{3l}.\end{gathered}\tag{19}$$

The equations (15), which satisfies conditions (18) have the form:

$$\begin{gathered}\varphi^{FL}(q,z,s)=C_1E_1(z);\quad \psi^{FL}(q,z,s)=C_2E_2(z);\\E_j(z)=e^{-k_j(q^2,s^2)z},\end{gathered}\tag{20}$$

where C_1 and C_2 - integration constants.

Substituting (20) into (16) and (17) leads to the following equalities for the images of displacements and stresses:

$$\begin{gathered}u_1^{FL}=-iqC_1E_1(z)+k_2(q^2,s^2)C_2E_2(z);\\u_3^{FL}=-k_1(q^2,s^2)C_1E_1(z)-iqC_2E_2(z);\end{gathered}\tag{21}$$

$$\begin{gathered}\eta^2\sigma_{11}^{FL}=(\chi\gamma_2^2s^2-2q^2)C_1E_1(z)-2iqk_2(q^2,s^2)C_2E_2(z);\\\eta^2\sigma_{13}^{FL}=2iqk_1(q^2,s^2)C_1E_1(z)-(\gamma_2^2s^2+2q^2)C_2E_2(z);\\\eta^2\sigma_{33}^{FL}=(\gamma_2^2s^2+2q^2)C_1E_1(z)+2iqk_2(q^2,s^2)C_2E_2(z).\end{gathered}\tag{22}$$

Substituting these equalities into the boundary conditions (19) leads to a system of linear algebraic equations for C_1 and C_2:

$$\begin{pmatrix}-iqa_{11}(q^2,s^2) & a_{12}(q^2,s^2)\\ a_{21}(q^2,s^2) & -iqa_{22}(q^2,s^2)\end{pmatrix}\begin{pmatrix}C_1\\C_2\end{pmatrix}=\eta^2\begin{pmatrix}\delta_{1l}\\\delta_{3l}\end{pmatrix},\tag{23}$$

where

$$\begin{gathered}a_{11}(q,s)=\alpha_1\eta^2-2\beta_1k_1(q,s);\\a_{12}(q,s)=\alpha_1\eta^2k_2(q,s)-\beta_1(\gamma_2^2s+2q);\\a_{21}(q,s)=-\alpha_3\eta^2k_1(q,s)+\beta_3(\gamma_2^2s+2q);\\a_{22}(q,s)=\alpha_3\eta^2-2\beta_3k_2(q,s).\end{gathered}\tag{24}$$

The solution to the system of equations (23) has the form:

$$C_1=\frac{\eta^2D_1(q,s)}{D(q^2,s^2)};\quad C_2=\frac{\eta^2D_2(q,s)}{D(q^2,s^2)},\tag{25}$$

where

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$$\begin{gathered}D(q,s)=-\alpha_1\alpha_3\eta^4R_1(q,s)-\left[\alpha_1\beta_3k_2(q,s)+\alpha_3\beta_1k_1(q,s)\right]\eta^2\gamma_2^2s+\beta_1\beta_3R_2(q,s);\\ D_1(q,s)=-iqa_{22}(q^2,s^2)\delta_{1l}-a_{12}(q^2,s^2)\delta_{3l},\\ D_2(q,s)=-a_{21}(q^2,s^2)\delta_{1l}-iqa_{11}(q^2,s^2)\delta_{3l},\\ R_1(q,s)=q-k_1(q,s)k_2(q,s),\\ R_2(q,s)=(\gamma_2^2s+2q)^2-4qk_1(q,s)k_2(q,s).\end{gathered}\tag{26}$$

Note that the function $R_2(q,s)$ is defined as follows:

$$\begin{gathered}R_2(q,s)=q^2R_0\left(-\gamma_2^2\frac{s}{q}\right);\\ R_0(\xi)=(2-\xi)^2-4\sqrt{1-\frac{\xi}{\eta^2}}\sqrt{1-\xi},\quad \xi=\frac{c^2}{c_2^2}.\end{gathered}\tag{27}$$

Taking into account (25), from (21) and (22) we obtain images of influence functions:

$$\begin{gathered}G_{1l}^{FL}=\frac{\eta^2}{D(q^2,s^2)}\Big\{-\left[q^2a_{22}(q^2,s^2)E_1(z)+k_2(q^2,s^2)a_{21}(q^2,s^2)E_2(z)\right]\delta_{1l}+\\ +iq\left[a_{12}(q^2,s^2)E_1(z)-k_2(q^2,s^2)a_{11}(q^2,s^2)E_2(z)\right]\delta_{3l}\Big\};\\ G_{3l}^{FL}=\frac{\eta^2}{D(q^2,s^2)}\Big\{\left[k_1(q^2,s^2)a_{22}(q^2,s^2)E_1(z)+a_{21}(q^2,s^2)E_2(z)\right]\delta_{1l}+\\ +\left[k_1(q^2,s^2)a_{12}(q^2,s^2)E_1(z)-q^2a_{11}(q^2,s^2)E_2(z)\right]\delta_{3l}\Big\};\end{gathered}\tag{28}$$

$$\begin{gathered}\Gamma_{11l}^{FL}=\frac{1}{D(q^2,s^2)}\Big\{iq\left[-(\chi\gamma_2^2s^2-2q^2)a_{22}(q^2,s^2)E_1(z)+2k_2(q^2,s^2)a_{21}(q^2,s^2)E_2(z)\right]\delta_{1l}-\\ -\left[(\chi\gamma_2^2s^2-2q^2)a_{12}(q^2,s^2)E_1(z)+2k_2(q^2,s^2)q^2a_{11}(q^2,s^2)E_2(z)\right]\delta_{3l}\Big\};\\ \Gamma_{13l}^{FL}=\Gamma_{31l}^{FL}=\frac{1}{D(q^2,s^2)}\Big\{\left[2k_1(q^2,s^2)q^2a_{22}(q^2,s^2)E_1(z)+(\gamma_2^2s^2+2q^2)a_{21}(q^2,s^2)E_2(z)\right]\delta_{1l}+\\ +iq\left[-2k_1(q^2,s^2)a_{12}(q^2,s^2)E_1(z)+(\gamma_2^2s^2+2q^2)a_{11}(q^2,s^2)E_2(z)\right]\delta_{3l}\Big\};\\ \Gamma_{33l}^{FL}=\frac{1}{D(q^2,s^2)}\Big\{-iq\left[(\gamma_2^2s^2+2q^2)a_{22}(q^2,s^2)E_1(z)+2k_2(q^2,s^2)a_{21}(q^2,s^2)E_2(z)\right]\delta_{1l}+\\ +\left[-(\gamma_2^2s^2+2q^2)a_{12}(q^2,s^2)E_1(z)+2k_2(q^2,s^2)q^2a_{11}(q^2,s^2)E_2(z)\right]\delta_{3l}\Big\}.\end{gathered}\tag{29}$$

These expressions are significantly simplified for the values of $G_{jl}^0(x,\tau)=G_{jl}\big|_{z=0}$ and $\Gamma_{jkl}^0(x,\tau)=\Gamma_{jkl}\big|_{z=0}$ of the influence functions on the boundary of the half-plane:

$$G_{1l}^{0FL}(q,s) = -\frac{\eta^2}{D(q^2,s^2)}\left\{\left[\alpha_3\eta^2 R_1(q^2,s^2) + \beta_3\gamma_2^2 k_2(q^2,s^2)s^2\right]\delta_{1l} + \right.$$
$$\left. + i\beta_1 q\left[\gamma_2^2 s^2 + 2R_1(q^2,s^2)\right]\delta_{3l}\right\};$$
$$G_{1l}^{0FL}(q,s) = \frac{\eta^2}{D(q^2,s^2)}\left\{\beta_3\left[\gamma_2^2 s^2 + 2R_1(q^2,s^2)\right]\delta_{1l} - \right.$$
$$\left. -\left[\alpha_1\eta^2 R_1(q^2,s^2) + \beta_1\gamma_2^2 k_1(q^2,s^2)s^2\right]\delta_{3l}\right\}; \tag{30}$$

$$\Gamma_{11l}^{0FL}(q,s) = \frac{1}{D(q^2,s^2)}\left\langle iq\left\{\alpha_3\eta^2\left[2R_1(q^2,s^2) - \chi\gamma_2^2 s^2\right] + \right.\right.$$
$$\left. + 2\beta_3(1+\chi)\gamma_2^2 s^2 k_2(q^2,s^2)\right\}\delta_{1l} - \left\{\alpha_1\eta^2\chi\gamma_2^2 s^2 k_2(q^2,s^2) - \right.$$
$$\left.\left. -\beta_1\left[(\chi\gamma_2^2 s^2 - 2q^2)(\gamma_2^2 s^2 + 2q^2) + 4q^2 k_1(q^2,s^2)k_2(q^2,s^2)\right]\right\}\delta_{3l}\right\rangle;$$
$$\Gamma_{13l}^{0FL}(q,s) = \Gamma_{31l}^{0FL}(q,s) = \frac{1}{D(q^2,s^2)}\left\{\left[-\alpha_3\eta^2\gamma_2^2 s^2 k_1(q^2,s^2) + \right.\right. \tag{31}$$
$$\left. \left. + \beta_3 R_2(q^2,s^2)\right]\delta_{1l} + i\alpha_1 q\eta^2\left[2R_1(q^2,s^2) + \gamma_2^2 s^2\right]\delta_{3l}\right\};$$
$$\Gamma_{33l}^{0FL}(q,s) = \frac{1}{D(q^2,s^2)}\left\{-i\alpha_3\eta^2 q\left[2R_1(q^2,s^2) + \gamma_2^2 s^2\right]\delta_{1l} + \right.$$
$$\left. + \left[-\alpha_1\eta^2\gamma_2^2 s^2 k_2(q^2,s^2) + \beta_1 R_2(q^2,s^2)\right]\delta_{3l}\right\}.$$

The calculation of the original influence functions depends significantly on the parameters in (12).

4. Calculation Examples

4.1. Example 1

Find the normal displacement $u_{30}(x,\tau) = u_3\big|_{z=0}$ on the boundary of the half-plane under the action of a tensile concentrated normal force on the surface of it, varying in time according to the law of the function Dirac-delta.

In this case, we have the second boundary value problem with boundary conditions (9) for $q_1 \equiv 0$ and $q_3 \equiv \delta(x)\delta(\tau)$. As follows from the definition of influence functions, for the desired displacement the following equality is true:

$$u_{30}(x,\tau) = G_{33}^0(x,\tau) \tag{32}$$

at $\alpha_1 = \alpha_3 = 0$ and $\beta_1 = \beta_3 = 1$.

Its image in accordance with (30) and (26) has the form:

$$u_{30}^{FL}(q,s) = -\frac{\eta^2\gamma_2^2 k_1(q^2,s^2)s^2}{R_2(q^2,s^2)}. \tag{33}$$

It is not possible to construct the original of this function by successive inversion of integral transformations. Therefore, we will use the joint inversion algorithm of the Fourier-Laplace transform. Carrying out the substitution $\lambda = q/s$ in (33), we obtain:

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$$u_{30}^{FL}(q,s) = g^L(s)h(\lambda),\ h(\lambda) = -\frac{\eta^2\gamma_2^2 k_1(\lambda^2,1)}{R_2(\lambda^2,1)},\ g^L(s) = \frac{1}{s}. \tag{34}$$

Considering that $g(\tau) = H(\tau)$ and $\dot{g}(\tau) = \delta(\tau)$, from the circulation table we find an analytical representation of the original:

$$\begin{gathered} \hat{u}_{30}(z,\tau) = -\frac{1}{2\pi} g'(\tau) * \varphi(z,\tau) = -\frac{1}{2\pi}\varphi(z,\tau); \\ z = x + iy; \quad \varphi(z,\tau) = h\left[\lambda(z,\tau)\right]\frac{\partial\lambda}{\partial\tau}, \end{gathered} \tag{35}$$

where

$$\begin{gathered} \lambda = \frac{\tau}{iz} = -\frac{\tau}{|z|^2}(y + ix); \\ \operatorname{Re}\lambda < 0 \quad (y \to +0); \quad \operatorname{Re}\lambda > 0 \quad y \to -0. \end{gathered} \tag{36}$$

The original itself is determined by the following equality:

$$u_{30}(x,\tau) = \lim_{y\to+0}\hat{u}_{30}(z,\tau) - \lim_{y\to-0}\hat{u}_{30}(z,\tau). \tag{37}$$

The function $\lambda(z,\tau)$ in (36) and the polynomial $\gamma_2^2 + 2\lambda^2(z,\tau)$ in the formula for $R_2(\lambda^2,1)$ in (26) are single-valued functions of the complex variable *z*, and the following relations are valid:

$$\begin{gathered} \lambda_0(x,\tau) = \lim_{y\to\pm0}\lambda(z,\tau) = -\frac{i\tau}{x}; \\ \lim_{y\to\pm0}\left[\gamma_2^2 + 2\lambda^2(z,\tau)\right] = \gamma_2^2 - 2\frac{\tau^2}{x^2}. \end{gathered} \tag{38}$$

We identify single-valued branches of square roots $k_j(\lambda^2,1)$ using sections of complex density λ along the imaginary axis. Taking into account (36) and (38), the limits of these roots are determined as follows:

$$\lim_{y\to\pm0} k_j(\lambda^2,1) = \begin{cases} \sqrt{\gamma_j^2 + \lambda_0^2} = \sqrt{\gamma_j^2 - \dfrac{\tau^2}{x^2}} & \text{при} \quad |\tau/x| < \gamma_i, \\ \pm i\operatorname{sign}x\sqrt{-\left(\gamma_j^2 + \lambda_0^2\right)} = \pm i\operatorname{sign}x\sqrt{\dfrac{\tau^2}{x^2} - \gamma_j^2} & \text{при} \quad |\tau/x| > \gamma_i. \end{cases} \tag{39}$$

Performing limit transitions in (37) and taking into account (38) and (39), we find the original of the desired displacement:

at $\tau/|x| < \gamma_1$:

$$u_{30} \equiv 0 \tag{40}$$

at $\gamma_1 \le \tau/|x| < \gamma_2$:

$$u_{30}(x,\tau)=u_{30}^{(1)}(x,\tau)=\frac{1}{2\pi ix}\left\{\lim_{y\to+0}h\left[\lambda(z,\tau)\right]-\lim_{y\to-0}h\left[\lambda(z,\tau)\right]\right\}=$$

$$=\frac{\eta^2\gamma_2^2\mathrm{i}\,\mathrm{sign}x}{2\pi ix}\sqrt{\frac{\tau^2}{x^2}-\gamma_1^2}\left\{\left[\left(\gamma_2^2-2\frac{\tau^2}{x^2}\right)^2+4\frac{\tau^2}{x^2}\mathrm{i}\,\mathrm{sign}x\sqrt{\gamma_2^2-\frac{\tau^2}{x^2}}\sqrt{\frac{\tau^2}{x^2}-\gamma_1^2}\right]^{-1}\right.$$

$$\left.+\left[\left(\gamma_2^2-2\frac{\tau^2}{x^2}\right)^2-4\frac{\tau^2}{x^2}\mathrm{i}\,\mathrm{sign}x\sqrt{\gamma_2^2-\frac{\tau^2}{x^2}}\sqrt{\frac{\tau^2}{x^2}-\gamma_1^2}\right]^{-1}\right\}= \tag{41}$$

$$=\frac{\eta^2\left(\gamma_2^2x^2-2\tau^2\right)^2}{\pi P_3(x^2,\tau^2)}\sqrt{\tau^2-\gamma_1^2x^2}.$$

at $\tau/|x|\ge\gamma_2$:

$$u_{30}(x,\tau)=u_{30}^{(3)}(x,\tau)=\frac{\eta^2\gamma_2^2x^2\sqrt{\tau^2-\gamma_1^2x^2}}{\pi R_{21}(x^2,\tau^2)}=\frac{\eta^2R_{22}(x^2,\tau^2)}{\pi P_3(x^2,\tau^2)}\sqrt{\tau^2-\gamma_1^2x^2} \tag{42}$$

The functions in (41) and (42) are related to each other and are defined by the following functions as follows:

$$\begin{gathered}R_{21}(x,\tau)=\left(\gamma_2^2x-2\tau\right)^2-4\tau\sqrt{\tau-\gamma_2^2x}\sqrt{\tau-\gamma_1^2x}=\tau^2R_0\left(\gamma_2^2x/\tau\right);\\ R_{22}(x,\tau)=\left(\gamma_2^2x-2\tau\right)^2-4\tau\sqrt{\tau-\gamma_2^2x}\sqrt{\tau-\gamma_1^2x}=\tau^2\tilde{R}_0\left(\gamma_2^2x/\tau\right);\\ R_{21}(x,\tau)R_{22}(x,\tau)=\left(\gamma_2^2x-2\tau\right)^4-16\tau^2\left(\tau-\gamma_2^2x\right)\left(\tau-\gamma_1^2x\right)=\gamma_2^2xP_3(x,\tau);\\ P_3(x,\tau)=\tau^3P_{32}\left(\gamma_2^2x/\tau\right)=\gamma_2^6x^3-8\gamma_2^4x^2\tau+8\left(2+\chi\right)\gamma_2^2x\tau^2-8\left(1+\chi\right)\tau^3\end{gathered} \tag{43}$$

Formulas (40)-(42) finally lead to the following result:

$$u_{30}(x,\tau)=\sum_{k=1}^{2}u_{30}^{(k)}(x,\tau)H\left(\tau-\gamma_k|x|\right) \tag{44}$$

where

$$u_{30}^{(2)}=u_{30}^{(3)}-u_{30}^{(1)}=\frac{4\eta^2\tau^2\left(\tau^2-\gamma_1^2x^2\right)}{\pi P_3\left(x^2,\tau^2\right)}\sqrt{\tau^2-\gamma_2^2x^2} \tag{45}$$

In Figure 1a, a space-time graph of the solution to the plane problem Lamb is presented for $\gamma_1=1$ and $\gamma_2=\eta=1{,}871$. The wave Rayleigh fronts are clearly visible on it. Figure 1b demonstrates the dependence on time and spatial coordinates of the regular part of the solution $\tilde{u}_{30}=\left(x^2-c_R^2\tau^2\right)u_{30}$. At the fronts of the wave Rayleigh, the values of this function are finite, and its derivative is infinite. The structure of this graph is explained by its sections with planes $\tau=0{,}3$; $\tau=0{,}9$ and $x=0{,}2$; $x=0{,}6$ shown respectively in Figure 1d and 1e.

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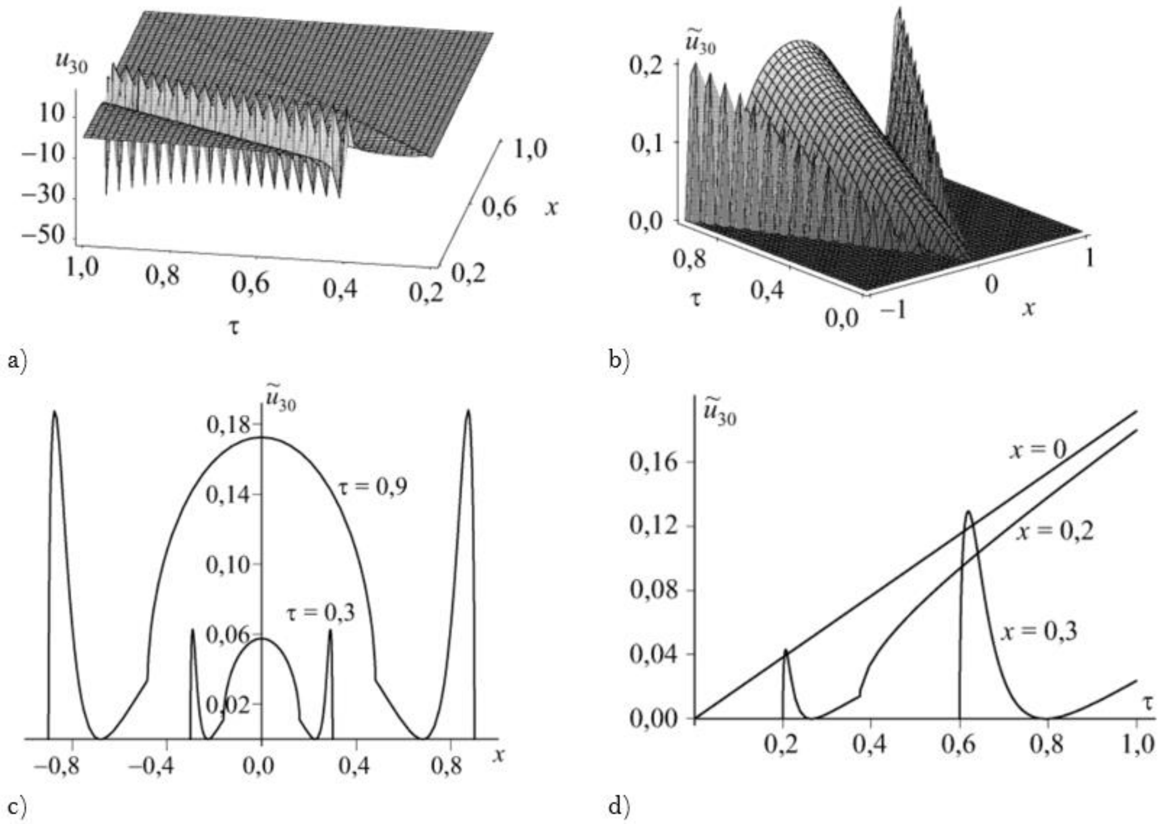

Figure 1.
Spatio-temporal graph of normal displacement on the boundary of a half-plane.

4.2. Example 2

We will find the normal stress $\sigma(x,\tau)=\sigma_{33}\big|_{z=0}$ on the boundary of the half-plane in the absence of tangential stress and a given concentrated normal surface displacement, which varies with time according to the law τ_+.

In this case, the boundary conditions are mixed excitation of the second type:

$$q_1 \equiv 0 \text{ và } q_3 \equiv \delta(x)\tau_+. \tag{46}$$

As follows from (14) and (46), the definition of influence functions and the properties of the delta-function, we have the following equality for the desired stress:

$$\sigma(x,\tau)=\Gamma^0_{333}(x,\tau)*\tau_+, \tag{47}$$

with $\alpha_1=\beta_3=0$ and $\beta_1=\alpha_3=1$.

The image of the influence function $\Gamma^0_{333}(x,\tau)$ in accordance with (29) and (26) has the form:

$$\Gamma^{0FL}_{333}(q,s)=\frac{1}{G^{0FL}_{33}(q,s)}=-\frac{R_2(q^2,s^2)}{\eta^2\gamma_2^2 s^2 k_1(q^2,s^2)}. \tag{48}$$

Then from (47) and (48), taking into account the properties of the Laplace transform, we obtain:

$$\sigma^{FL}(q,s)=-\frac{R_2(q^2,s^2)}{\eta^2\gamma_2^2 s^4 k_1(q^2,s^2)}. \tag{49}$$

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The original of this function can be found using the joint inversion algorithm of the Fourier-Laplace transform, based on analytical representations of images. Here we use the notations for the functions $k_j(q,s)$ in (15) and $R_2(q,s)$ in (26). First, we represent σ^{FL} as:

$$\sigma^{FL}(q,s) = -\frac{1}{\eta^2\gamma_2^2}\left[\left(\gamma_2^2 + 2\frac{q^2}{s^2}\right)^2 \frac{1}{\sqrt{\gamma_1^2 s^2 + q^2}} - 4\frac{q^2}{s^2}\left(\gamma_2^2 + \frac{q^2}{s^2}\right)\frac{1}{\sqrt{\gamma_2^2 s^2 + q^2}}\right]. \qquad (50)$$

Next, using the inverse Laplace transform we get:

$$\left[\left(\gamma_j^2 s^2 + q^2\right)^{-1/2}\right]^{F^{-1}L^{-1}} = \frac{1}{\pi}\left[K_0(\gamma_j |x| s)\right]^{L^{-1}} = \frac{1}{\pi}(\tau^2 - \gamma_j^2 x^2)_+^{-1/2}. \qquad (51)$$

We note that the function $\left(\gamma_j^2 s^2 + q^2\right)^{-1/2}$ is homogeneous of degree (-1) and using the inverse transformation we get:

$$\left[\left(\gamma_2^2 + 2\frac{q^2}{s^2}\right)^2 \frac{1}{\sqrt{\gamma_1^2 s^2 + q^2}}\right]^{F^{-1}L^{-1}} = \frac{1}{\pi}\left(\gamma_2^2 - 2\frac{\tau^2}{x^2}\right)^2 (\tau^2 - \gamma_1^2 x^2)_+^{-1/2};$$
$$\left[\frac{q^2}{s^2}\left(\gamma_2^2 + \frac{q^2}{s^2}\right)\frac{1}{\sqrt{\gamma_2^2 s^2 + q^2}}\right]^{F^{-1}L^{-1}} = -\frac{\tau^2}{\pi x^2}\left(\gamma_2^2 - \frac{\tau^2}{x^2}\right)(\tau^2 - \gamma_2^2 x^2)_+^{-1/2}. \qquad (52)$$

Using these equalities, from (50) we finally obtain the following expression for the desired stress:

$$\sigma(x,\tau) = \sum_{k=1}^{2}\sigma^{(k)}(x,\tau)H(\tau - \gamma_k |x|);$$
$$\sigma^{(1)}(x,\tau) = -\frac{(\gamma_2^2 x^2 - 2\tau^2)^2}{\pi\eta^2\gamma_2^2 x^4\sqrt{\tau^2 - \gamma_1^2 x^2}}; \qquad (53)$$
$$\sigma^{(2)}(x,\tau) = -\frac{4\tau^2}{\pi\eta^2\gamma_2^2 x^4}\sqrt{\tau^2 - \gamma_2^2 x^2}.$$

In Figure 2a the space-time graph constructed at $\gamma_1 = 1$ and $\gamma_2 = \eta = 1{,}871$ for the normal stress in this problem is presented. Due to the parity of the function, its part at $x \geq 0$ is depicted. In this figure, for clarity, due to their large magnitude, the absolute values of the stress in the vicinity of straight line $\tau = x$ are not indicated. Regular part $-x^4\sigma(x,\tau)$ is shown in Figure 2b.

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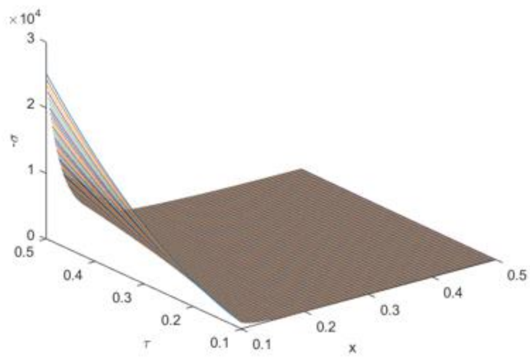
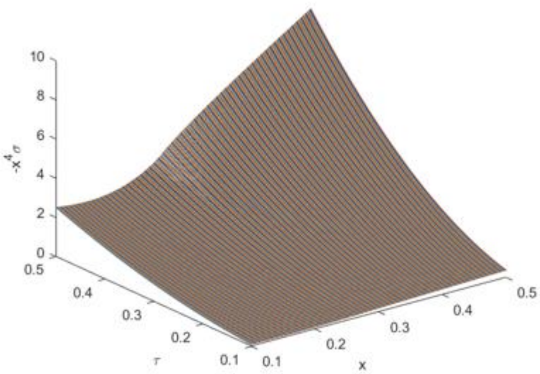


a)



b)

Figure 2.
Space-time graph for normal stress at the boundary of a half-plane.

5. Conclusion

The suggested technique aims to solve issues related to the propagation of nonstationary two-dimensional boundary disturbances in a half-plane. The issue may be solved by obtaining explicit integral formulae, which allow for the determination of unknown displacements and stresses. The joint inversion procedure of the Fourier-Laplace transform is used to seek for the original picture function. Asymptotic representations of stresses and displacements around the point where the boundary conditions separate. We computed two issues that correspond to two different forms of boundary value problems: kinematic excitation and mixed excitation. A space-time graph was generated, displaying the Rayleigh wavefronts with clarity. This approach allows us to address issues involving different potential boundary conditions about the spread of stationary two-dimensional boundary disturbances in a half-plane.

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References

[1] H. Lamb, "On the propagation of tremors over the surface of an elastic solid," *Proc. R. Soc. London*, vol. 72, no. 477–486, pp. 128–130, 1904, doi: 10.1098/rspl.1903.0029.

[2] H. De-sui, "An application of Ungar's differential transform to elastodynamics," *Appl. Math. Mech.*, vol. 10, no. 7, pp. 675–678, 1989, doi: 10.1007/BF02115800.

[3] D. Kosloff, M. Reshef, and D. Loewenthal, "Elastic wave calculations by the Fourier method," *Bull. Seismol. Soc. Am.*, vol. 74, no. 3, pp. 875–891, 1984, doi: 10.1785/bssa0740030875.

[4] P. G. Richards, "Elementary solutions to Lamb's problem for a point source and their relevance to three-dimensional studies of spontaneous crack propagation," *Bull. Seismol. Soc. Am.*, vol. 69, no. 4, pp. 947–956, 1979, doi: 10.1785/bssa0690040947.

[5] B. Steinfeld, H. Takemiya, and H. Antes, "Analysis of Transient 3D Wave Propagation in an Elastic Halfspace: The classical Approach and the Direct Boundary Element Method," 1994.

[6] Y. A. Melnikov, "Influence functions of a point source for perforated compound plates with facial convection," *J. Eng. Math.*, vol. 49, no. 3, pp. 253–270, 2004, doi: 10.1023/B:ENGI.0000031187.96637.ea.

[7] C. M. Churchman, A. M. Korsunsky, and D. A. Hills, "The edge dislocation in a three-quarter plane. Part I: Influence functions," *Eur. J. Mech. A/Solids*, vol. 25, no. 1, pp. 42–50, 2006, doi: 10.1016/j.euromechsol.2005.06.010.

[8] D. V. Tarlakovskii and G. V. Fedotenkov, "Nonstationary 3D motion of an elastic spherical shell," *Mech. Solids*, vol. 50, no. 2, pp. 208–217, 2015, doi: 10.3103/S0025654415020107.

[9] L. T. Tuan, N. Van Dung, P. Van Minh, B. D. Tan, D. Van Thom, and A. M. Zenkour, "Analysis of the Stress–Strain State of the Elastic Moment Medium When a Spherical Cavity Diffracts the Wave," *J. Vib. Eng. Technol.*, 2023, doi: 10.1007/s42417-023-01155-5.

[10] L. T. Tuan, N. T. Dung, D. Van Thom, P. Van Minh, and A. M. Zenkour, "Propagation of non-stationary kinematic disturbances from a spherical cavity in the pseudo-elastic cosserat medium," *Eur. Phys. J. Plus*, vol. 136, no. 12, 2021, doi: 10.1140/epjp/s13360-021-02191-4.

[11] E. Y. Mikhailova and G. V. Fedotenkov, "Nonstationary axisymmetric problem of the impact of a spherical shell on an

elastic half-space (initial stage of interaction)," *Mech. Solids*, vol. 46, no. 2, pp. 239–247, 2011, doi: 10.3103/S0025654411020129.
[12] S. V. Shmegera, "The initial boundary-value mixed problems for elastic half-plane with the conditions of contact friction," *Int. J. Solids Struct.*, vol. 37, no. 43, pp. 6277–6296, 2000, doi: 10.1016/S0020-7683(99)00295-4.
[13] Y. M. Suvorov, D. V. Tarlakovskii, and G. V. Fedotenkov, "The plane problem of the impact of a rigid body on a half-space modelled by a Cosserat medium," *J. Appl. Math. Mech.*, vol. 76, no. 5, pp. 511–518, 2012, doi: 10.1016/j.jappmathmech.2012.11.015.
[14] D. V. Tarlakovskii and G. V. Fedotenkov, "Two-dimensional nonstationary contact of elastic cylindrical or spherical shells," *J. Mach. Manuf. Reliab.*, vol. 43, no. 2, pp. 145–152, 2014, doi: 10.3103/S1052618814010178.
[15] L. A. Igumnov, A. S. Okonechnikov, D. V. Tarlakovskii, and G. V. Fedotenkov, "Plane Nonstationary Problem of Motion of the Surface Load Over an Elastic Half Space," *J. Math. Sci. (United States)*, vol. 203, no. 2, pp. 193–201, 2014, doi: 10.1007/s10958-014-2100-z.
[16] L. A. Igumnov, A. A. Belov, and A. N. Petrov, "Boundary-Element Modeling of the Dynamics of Elastic and Viscoelastic Bodies and Media," *Adv. Mater. - Stud. Appl.*, pp. 301–317, 2015.
[17] L. A. Igumnov, I. P. Markov, and A. V. Amenitsky, "A three-dimensional boundary element approach for transient anisotropic viscoelastic problems," *Key Eng. Mater.*, vol. 685, pp. 267–271, 2016, doi: 10.4028/www.scientific.net/KEM.685.267.
[18] Poruchikov VB, *Theoretical methods of elastic dynamics.* 1986.
[19] D. V. Quang, N. D. Tran, D. T. Luat, T. V. Do, "Static analysis and boundary effect of FG-CNTRC cylindrical shells with various boundary conditions using quasi-3D shear and normal deformations theory," *Struct.*, vol. 44, pp. 828–850, 2022, doi: 10.1016/j.istruc.2022.08.039.
[20] P. V. Vinh, N. T. Dung, N. C. Tho, T. V. Do, N. T. Trung, L. K. Hoa, "Modified single variable shear deformation plate theory for free vibration analysis of rectangular FGM plates," *Strut.*, vol. 29, pp. 1435-1444, 2021, doi: 10.1016/j.istruc.2020.12.027.
[21] D. H. Duc, T. V. Do, P. M. Phuc, "Buckling Analysis of Variable Thickness Cracked Nanoplatesconsiderting the Flexoelectric Effect," *Trans. Comm. Scie. J.*, vol. 73, no. 5, pp. 470-485, 2022, doi: 10.47869/tcsj.73.5.3
[22] N.C. Tho, N.T. Ta, T. V. Do, "New numerical results from simulations of beams and space frame systems with a tuned mass damper," *Materials*, vol. 12, no. 8, 1329, 2019, doi: 10.3390/ma12081329
[23] D. M. Tien, T. V. Do, N.T.H. Van, A. Tounsi, P.V. Minh, D.N. Mai, "Buckling and forced oscillation of organic nanoplates taking the structural drag coefficient into account," *Comp. Concr.*, vol. 6, no. 32, pp. 553-565, 2024, doi:10.12989/cac.2023.32.6.553
[24] P. H. Cong, D. H. Duc, D. V. Thom, "Phase field model for fracture based on modified couple stress," *Eng. Fract. Mech.*, vol. 269, 108534, 2022, doi: 10.1016/j.engfracmech.2022.108534.
[25] T. D. Minh, V. T. Do, M. P. Van, H. P. Huy, "Bending and buckling responses of organic nanoplates considering the size effect," *Trans. Comm. Scie. J.*, vol. 75, no. 7, pp. 2015-2029, 2022, doi: 10.47869/tcsj.75.7.1.
[26] L. T. Son, V. M. Chinh, "Bending and buckling responses of nanoplates under the compresstion a part of edges," *Trans. Comm. Scie. J.*, vol. 75, no. 9, pp. 2289-2302, 2024, doi: 10.47869/tcsj.75.9.5
[27] T. V. Toan, T. T. H. Huyen, D. V. Doan, "Vibration and buckling analysis of nanoplates resting on variable elastic foundations," *Trans. Comm. Scie. J.*, vol. 75, no. 9, pp. 2238-2251, 2024, doi: 10.47869/tcsj.75.9.1