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The excision theory for homology theory through A_∞-algebras

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Abstract: This paper investigated A_∞-algebras, which are generalizations of associative algebras that incorporate higher homotopy structures. We began by revisiting the fundamental definitions and properties of A_∞-algebras and their associated homological theories, providing a solid foundation for understanding these complex structures. The study included an in-depth analysis of simplicial homology as it relates to A_∞-algebras, focusing on significant results, particularly those concerning excision theory. In this context, we introduced new insights into the relationship between bar homology and simplicial homology, presenting a precise sequence elucidating the interaction between these two homological structures. Within this framework, we provided proofs for key results, such as the quantitative coherence of certain maps and the interchanging diagram that connects different homological categories. We address the specific failure of excision properties and its implications for long exact sequences in both homological and homotopical contexts. This paper offered a comprehensive overview of current developments in A_∞-algebra theory and simplicial cohomology, highlighting classical and contemporary insights into these sophisticated mathematical structures. By presenting detailed definitions, examples, and theorems, we strive to contribute to a deeper understanding of homology within the framework of advanced algebraic systems. Our analysis sheds light on existing theories and paves the way for future research in the field, providing a valuable resource for mathematicians interested in the interplay between algebra and topology.
Keywords: *A_∞-algebras, Excision, Hochschild, Homology.*

1. Introduction

As demonstrated by Riemann's solutions to problems involving surface connections, homology is pivotal in mathematical investigations. Green's theorem, which relates line integrals over a closed curve to double integrals over the enclosed plane region, underpins this principle. The theorem implies that certain integrals will yield identical values for any two homologous curves, an idea that permeates classical vector spaces and theoretical physics.

Simplicial homology, a branch of algebraic topology, addresses the study of homology theory in associative algebras over a field. Gerhard Hochschild introduced this theory [1], initially focusing on algebras over a field. Henri Cartan and Samuel Eilenberg [2] later extended this work to more generalized rings.

In [3], Stasheff developed the concepts of $\mathcal{A}_\infty$-spaces and $\mathcal{A}_\infty$-algebras during his study of topological spaces. These ideas emerged as generalizations of topological groups while maintaining continuous, albeit non-associative, multiplication. $\mathcal{A}_\infty$-algebras are essentially chain complexes with homotopy associative products, satisfying higher homotopy associativity conditions. A prime example of an $\mathcal{A}_\infty$-algebra is the singular chain complex $\mathcal{C}_\bullet(\mathcal{X})$, which illustrates their homotopy-invariant nature.

Research into $\mathcal{A}_\infty$-algebras has advanced significantly due to contributions from mathematicians such as Kadeishvili [4], Smirnov [5], and Prouté [6]. J. Huebschmann highlighted the relevance of homological perturbation theory and $\mathcal{A}_\infty$-structures in homological algebra, especially within

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topological contexts [7]. Further advancements were made by John D. S. Jones and E. Getzler [8]. Kenji Fukaya's exploration of $\mathcal{A}_\infty$-categories [9] and Kontsevich's influential 1994 lecture on categorical mirror symmetry contributed significantly to the field's development [10].

In [11], Keller integrated $\mathcal{A}_\infty$-language into ring theory and representation theory, showing that the derived category of any Grothendieck category with a compact generator is equivalent to the derived category of an $\mathcal{A}_\infty$-algebra. Additional studies by P. Seidel on $\mathcal{A}_\infty$-structures concerning Lefschetz fibrations [12], and research by Alaa H. and Y. Gouda on simplicial cohomology for $\mathcal{A}_\infty$-algebras [13,14] expanded our understanding of these structures.

The lack of excision in specific contexts leads to long exact sequences in homological categories but not in homotopic categories. This paper presents various definitions of homology for $\mathcal{A}_\infty$-algebras, examining the simplicial homology of these structures and providing reliable results for their excision theory. We explored the excision theory of simplicial homology for $\mathcal{A}_\infty$-algebras, establishing the relationship between bar homology $\mathcal{H}B_n(\mathcal{J})$ and simplicial homology $\mathcal{H}\mathcal{H}_n(\mathcal{J})$ through an exact sequence:

$$\dots \longleftarrow H_{n-1}(\mathcal{J}) \longleftarrow \mathcal{H}B_{n-1}(\mathcal{J}) \longleftarrow \mathcal{H}\mathcal{H}_n(\mathcal{J}) \longleftarrow H_n(\mathcal{J}) \longleftarrow \mathcal{H}B_n(\mathcal{J}) \longleftarrow \mathcal{H}\mathcal{H}_{n+1}(\mathcal{J}) \longleftarrow \dots..$$

Additionally, we proved that the quasi-isomorphisms for the following maps:

$i: \left(\mathcal{R} \otimes \mathcal{J}^{\otimes *}, \rho_*\right) \otimes \mathcal{G} \hookrightarrow \left(\mathcal{R} \otimes \mathcal{A}^{\otimes *}, \rho_*\right) \otimes \mathcal{G}$,

$i': \left(\mathcal{R} \otimes \mathcal{J}^{\otimes *}, \rho'_*\right) \otimes \mathcal{G} \hookrightarrow \left(\mathcal{R} \otimes \mathcal{A}^{\otimes *}, \rho'_*\right) \otimes \mathcal{G}$,

are quasi-isomorphisms, as are the two maps:

$\pi: \left(\mathcal{B} \otimes \mathcal{A}^{\otimes *}, \rho_*\right) \otimes \mathcal{G} \longrightarrow \left(\mathcal{B} \otimes \mathcal{B}^{\otimes *}, \rho_*\right) \otimes \mathcal{G}$,

$\pi': \left(\mathcal{B} \otimes \mathcal{A}^{\otimes *}, \rho'_*\right) \otimes \mathcal{G} \longrightarrow \left(\mathcal{B} \otimes \mathcal{B}^{\otimes *}, \rho'_*\right) \otimes \mathcal{G}$,

These results enable us to construct a commutative diagram, elucidating the intricate relationships between these homological structures:

$$\begin{array}{ccccccccc}
0 & \to & (\mathcal{J} \otimes \mathcal{A}^{\otimes *}, \rho_*) \otimes \mathcal{G} & \to & (\mathcal{A} \otimes \mathcal{A}^{\otimes *}, \rho_*) \otimes \mathcal{G} & \to & (\mathcal{B} \otimes \mathcal{A}^{\otimes *}, \rho_*) \otimes \mathcal{G} & \to & 0 \\
 & & |j & & |= & & |\pi_1 & & \\
0 & \to & ker\,(\pi) & \to & (\mathcal{A} \otimes \mathcal{A}^{\otimes *}, \rho_*) \otimes \mathcal{G} & \xrightarrow{\pi} & (\mathcal{B} \otimes \mathcal{A}^{\otimes *}, \rho_*) \otimes \mathcal{G} & \to & 0
\end{array}.$$

2. Homology theory of $\mathcal{A}_\infty$-algebras

In this section, we will explore the fundamentals and definitions related to the homology theory of $\mathcal{A}_\infty$-algebras. We begin by introducing the basic definitions and concepts of $\mathcal{A}_\infty$-algebras. Following this, we will present the simple homology of $\mathcal{A}_\infty$-algebras.

Firstly, we define an algebra over the field $\mathcal{R}$, focusing on properties.

2.1. Definition [2]

Over the field $\mathcal{R}$, algebra is represented by the linear vector space $\mathcal{X}$, which has the multiplication function $\mathcal{T}: \mathcal{X} \times \mathcal{X} \to \mathcal{X}$, $(v, u) \mapsto vu$, that which $\mathcal{T}$ is distributed and linear in the two variables. That applies for all $v, u, w \in \mathcal{X}, \alpha \in \mathcal{R}$

- $w(v + u) = wu + wu$.
- $(v + u)w = vw + uw$.
- $\alpha(vu) = (\alpha v)u = v(\alpha u)$.

Following, we define graded vector spaces where we specify the vector space $\mathcal{X}$ and related graded vector spaces $\mathcal{X}_\iota$. We describe how to handle homogeneous elements in these spaces.

2.2. Definition [3]

Suppose $\mathcal{J}$ denotes the set of index. The vector space $\mathcal{X}$ that has the grade $\mathcal{J}$, known as the $\mathcal{J}$-graded vector space, takes the following form $\mathcal{X} = \bigoplus_{\iota \in \mathcal{J}} \mathcal{X}_\iota$, hence for each ι, and then $\mathcal{X}_\iota$ would be a vector

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space. The elements $x \in \mathcal{X}_\iota$ are hence known as homogeneous elements with degree ι and denoted by $deg\, x = \iota$ or $|x| = \iota$.

Next, we introduce the tensor product of vector spaces $\mathcal{X}$ and $\mathcal{Y}$ over a field $\mathfrak{F}$. We define the bilinear map and discuss the properties of these tensor spaces.

2.3. Definition [3]

If the vector spaces $\mathcal{X}$,$\mathcal{Y}$ over a field $\mathfrak{F}$ with elements $\{x_\iota\}_{\iota\in\mathcal{J}}$ and $\{y_\ell\}_{\ell\in\mathcal{L}}$; then, the tensor product $\mathcal{X} \otimes \mathcal{Y}$ is defined as a vector space over $\mathfrak{F}$ with the symbols $\{x_\iota \otimes y_\ell\,,\ \forall\iota \in \mathcal{J}\,,\ \ell \in \mathcal{L}\}$ as its basis. Additionally, we define the bilinear map $\mathcal{X} \times \mathcal{Y} \to \mathcal{X} \otimes \mathcal{Y}$ as the two vectors' combined tensor product $x = \sum_\iota a_\iota x_\iota$ and $y = \sum_\ell b_\ell y_\ell$, which is provided by:

$\mathcal{X} \otimes \mathcal{Y} = (\sum_\iota a_\iota x_\iota) \otimes (\sum_\ell b_\ell y_\ell) = \sum_{\iota,\ell} a_\iota b_\ell (x_\iota \otimes y_\ell)$.

In the following proposition, we will prove the existence of a unique linear map between the tensor products of vector spaces. We will demonstrate how to establish this map and its distinctive properties.

2.4. Proposition

Assume that $\mathcal{M}$ is the vector space. A unique linear map $\mathfrak{R}': \mathcal{X} \otimes \mathcal{Y} \to \mathcal{M}$ exists for the bilinear map $\mathfrak{R}: \mathcal{X} \times \mathcal{Y} \to \mathcal{M}$, such as $\mathfrak{R} = \mathfrak{R}' \circ \phi$ and ϕ is the normal incorporation of $\mathcal{X} \times \mathcal{Y}$ in $\mathcal{X} \otimes \mathcal{Y}$. Additionally, the universal characteristic is satisfied by the unique isomorphism in a vector space with a bilinear map.

Proof:

The relation $\mathfrak{R}'(x_\iota \otimes y_\ell) = \beta(x_\iota, y_\ell)$ is a basic $\mathcal{X} \otimes \mathcal{Y}$ in $\mathfrak{R}'$. This map is unique because it satisfies the requirement for $\mathfrak{R} = \mathfrak{R}' \circ \phi$. Now, let $\mathcal{N}$ be the vector space with the bilinear map $\omega: \mathcal{X} \times \mathcal{Y} \to \mathcal{N}$. So, for each bilinear map $\mathfrak{R}: \mathcal{X} \times \mathcal{Y} \to \mathcal{M}$, there is a unique linear map $\mathfrak{R}'$, for instance $\mathfrak{R} = \mathfrak{R}' \circ \phi$.

Suppose that $\mathcal{M} = \mathcal{X} \otimes \mathcal{Y}$ and $\mathfrak{R}$ is the map $(x, y) \mapsto x \otimes y$. Therefore, through the general property of $\mathcal{N}$, a unique linear map $\mathfrak{R}': \mathcal{N} \to \mathcal{X} \otimes \mathcal{Y}$ exists as $\mathfrak{R} = \mathfrak{R}' \circ \phi$. Likewise, through the general property of $\mathcal{X} \otimes \mathcal{Y}$, there exists the unique linear map $\mathcal{E}': \mathcal{X} \otimes \mathcal{Y} \to \mathcal{N}$ as $\omega = \mathcal{E}' \circ \phi$ so $\phi = (\mathfrak{R}' \circ \mathcal{E}') \circ \phi$.

Therefore, $\mathfrak{R}' \circ \mathcal{E}' = id$ and $\mathfrak{R}'$ is unique isomorphism $\mathfrak{R}': \mathcal{N} \xrightarrow{\sim} \mathcal{X} \otimes \mathcal{Y}$.

In the following, we define a graded algebra over a field $\mathscr{R}$ and how to deal with the resulting multiplicity of multiplication maps in this type of algebra.

2.5. Definition [14]

Suppose $\mathcal{M}$ is the graded vector space on the field $\mathcal{R}$, then the graded algebra on $\mathcal{R}$ is defined as the algebra $\mathcal{M}$ such as $\mathcal{M} = \bigoplus_{\iota \in \mathcal{J}} \mathcal{M}_\iota$, and the multiplication map for all $m, n \in \mathcal{M}$ is given by:

$deg\, mn = deg\, m + deg\, n$.

After that, we will introduce the construction of a tensor algebra from a vector space $\mathcal{M}$, explaining how this construction produces a graded algebra defined by the tensor product.

2.6. Definition [15]

If $\mathcal{R}$ is a field, then the vector space $\mathcal{M}$ provides the tensor algebra of $\mathcal{M}$ by:

$$T(\mathcal{M}) = \mathcal{R} \oplus \mathcal{M} \oplus \mathcal{M}^{\otimes 2} \oplus \mathcal{M}^{\otimes 3} \oplus \dots$$

The tensor product's presumed multiplication appears as a graded algebra.

In the following definition, we define linear morphisms between graded algebras using the tensor product and describe the degree-related properties of these morphisms.

2.7. Definition [15]

Suppose that $\mathcal{M} = \bigotimes_{n\in Z} \mathcal{M}_n$ is the $\mathbb{Z}$-graded vector space. Then, the homeomorphisms of vector spaces are linear maps $f: \mathcal{M}^{\otimes n} \to \mathcal{M}$. Since the degree of $f, |f|$ is given by ι, when $deg\, f(m) = (deg\, m) + \iota$ for all $m \in \mathcal{M}$. Obviously,
$|f(m_1 \otimes m_2 \otimes \cdots \otimes m_n)| = (|m_1| + |m_2| + \cdots + |m_n|) + \iota$.

We will provide an illustrative example of a graded algebra using a complex sequence over a specific field and explaine how to explore homology through this example.

2.8. Example

Assume the complex over a field known as:

$$V_\bullet : \dots \longrightarrow V_{n+1} \xrightarrow{d_{n+1}} V_n \xrightarrow{d_n} V_{n-1} \longrightarrow \dots ,$$

hence the modules V_n seem to be, in fact, vector spaces. So $V = \oplus_n V_n$ is known as the graded vector space, where $d: V^{\otimes 1} \to V$ have a degree -1.

Following, we define morphisms between $\mathcal{A}_\infty$-algebras and explain the conditions and properties necessary for these morphisms.

2.9. Definition [13]

By assuming that β, γ are two linear maps of the graded vector spaces $\mathcal{X}, \mathcal{Y}$ respectively, such that $\beta: \mathcal{X} \to \mathcal{X}$, $\gamma: \mathcal{Y} \to \mathcal{Y}$. Next, the linear maps' tensor product β and γ is denoted by:

$$\beta \otimes \gamma : \mathcal{X} \otimes \mathcal{Y} \to \mathcal{X} \otimes \mathcal{Y}$$

$$(x \otimes y) \mapsto (-1)^{|\gamma||y|}(\beta(x) \otimes \gamma(y)). \qquad (1)$$

Remind that the degree of $(\beta \otimes \gamma)$ is $|\beta| + |\gamma|$.

Remark: The Koszul sign rule states that if two symbols' positions s and r are switched, the outcome is multiplied by $(-1)^{|s||r|}$, which is the actual cause of the change in sign in the previous expression. In the expression above, it is applied as $(\beta \otimes \gamma)(x \otimes y) = (-1)^{|\gamma||y|}(\beta(x) \otimes \gamma(y))$, where the morphisms acting on the elements are represented by the symbols x and y, which are swapped.

In the following, we describe the concept of differential graded algebras (DGAs) and how to define them using the chain complex maps and Leibniz rules.

2.10. Definition [15]

The differential graded algebra (*DGA*) is the complex $\mathcal{M}$ with the degree $+1$ chain map $d: \mathcal{M} \otimes \mathcal{M} \to \mathcal{M}$ so that $d^2 = 0$, which is unital and associative. The differential graded algebra is simply a differential d and a graded module equivalent.

The graded Leibniz rule for all $m, n \in \mathcal{M}$ is denoted by:

$$d(mn) = d(m)n + (-1)^{|m|} d(n). \qquad (2)$$

Remark:

(1) If $mn = (-1)^{|m||n|} nm$, for all $m, n \in \mathcal{M}$, then the differential graded algebra $\mathcal{M}$ is regarded as commutative.

If $1/2 \in k$, it is implied that $m^2 = 0$ holds true if m would have an odd degree.

When $\mathcal{M}$ is the commutative graded algebra, therefore the left $\mathcal{M}$-module, and automatically R is a right $\mathcal{M}$-module, by the formula $rm = (-1)^{|r||m|} mr$.

(2) Similar definitions are provided for the terms differential graded algebra and derivation.

Here, we will explain how to handle left models of differential graded algebras and use rules for dealing with derivatives in these models.

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2.11. Definition [16]

Assume that the differential graded algebra is $(\mathcal{M}, d)$. In addition, let R be the left graded $\mathcal{M}$-module and d_R be the differential of R, so the complex $(R, d_R)$and the left multiplication $\mathcal{M} \otimes R \to R$ combined to create the left differential graded $\mathcal{M}$-module such that the Leibniz rule satisfies for all $m \in \mathcal{M}, r \in R$: $d_R(mr) = d(m)r + (-1)|m|md_R(r)$.
An $\mathcal{M}$-module with differential grading is merely a complex. Similar terms are used to define a right differential graded $\mathcal{M}$-module.

After that, we define graded spaces using morphisms and methods to determine derivatives within these graded spaces.

2.12. Definition [16]

Let $Hom_{\mathcal{M}}(R, S)$ be a graded vector space that contains $\mathcal{M}$-homomorphisms as of R to S. Graded modules: $Hom_{\mathcal{M}}(R, S) := \bigoplus_{\iota \in \mathbb{Z}} Hom_{\mathcal{M}}^{Gr}(R, S)^{\iota}$,
using the differential d_{Hom} determined to be:
$d_{Hom}(\hbar) = d_S \circ \hbar - (-1)^{|\hbar|} \hbar \circ d_R, \quad \forall\, \hbar \in Hom_{\mathcal{M}}(R, S)$.
In particular, differential graded algebras are used to create the graded vector space $Hom_{\mathcal{M}}(R, R)$, where $Hom_{\mathcal{M}}(L, R)$ is a differential-graded module over $Hom_{\mathcal{M}}(R, R)$ differential-graded algebras.

Next, we will describe how to handle the tensor product of graded spaces and determine the graded map's properties in this context.

2.13. Definition [17]

By the differential $d_{\otimes}$ and the complex $\mathcal{M}$, we can define the graded vector space for a tensor product $R \otimes_{\mathcal{M}} S$ over $\mathcal{M}$ as:
$d_{\otimes}(r \otimes_{\mathcal{M}} s) = d_R(r) \otimes_{\mathcal{M}} s + (-1)^{|r|} r \otimes_{\mathcal{M}} d_S(s)$.
There is an adjoint property between $Hom_{\mathcal{M}}$ and $\otimes_{\mathcal{M}}$:
$Hom_{\mathcal{M}}(L \otimes_{\mathcal{M}} R, S) \cong Hom_{\mathcal{M}}(L, Hom_{\mathcal{M}}(R, S))$.

Here, we will define $\mathcal{A}_{\infty}$-algebras and how to handle their fundamental properties, such as the Stasheff identity.

2.14. Definition [18]

A vector space with $\mathbb{Z}$ grades is an $\mathcal{A}_{\infty}$-algebras on a field M such that: $\mathcal{A} = \bigoplus_{\wp \in \mathbb{Z}} \mathcal{A}^{\mathcal{P}}$, supplied with graded maps that are homogenous M-linear mappings;
$r_n : \mathcal{A}^{\otimes n} \to \mathcal{A}, \qquad n \geq 1$.
of $(|r_n| = 2 - n)$-degree, fulfilling the subsequent requirements of Stasheff identities: $\forall\, n \in \mathbb{N}$, $(SL(n))$

$$\sum (-1)^{m+st} r_{q}\left(id^{\otimes m} \otimes rs \otimes id^{\otimes t}\right) = 0 \qquad (3)$$

where the total is applied to all decompositions $n = m + s + t$, $m, t \geq 0$ and $s \geq 1$, and $q = m + 1 + t$. Thus, id refers to the identification map of $\mathcal{A}$. Because of the Koszul sign rule, additional signs occur when these formulas get used on elements. $\mathcal{A}_{\infty}$-algebras are known as strongly homotopy associative algebras. For example:

1) $SL(1)$ means that: $r_1 \circ r_1 = 0$.
   where r_1 has a degree of 1, which means that r_1 is a derivative of $\mathcal{A}$.
2) $SL(2)$ says that r_1 is a derivation for r_2, such that:

$$r_1 \circ r_2 = r_2 \circ (\, r_1 \otimes id + id \otimes r_1)$$

The degree of r_2 is zero.
3) $SL(3)$ indicates that r_2 is associative till the homotopy r_3 can be re-written as:

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$r_2 \circ (id \otimes r_2 - r_2 \otimes id) = \partial(r_3)$
$$= r_1 \circ r_3 + r_3 \circ (r_1 \otimes id \otimes id + id \otimes r_1 \otimes id + id \otimes id \otimes r_1)$$
where ∂ is the differential of $Hom(\mathcal{A}^{\otimes 3}, \mathcal{A})$ induced by r_1.
4) Pentagonal homotopy associative algebra $(\mathcal{A}, r_1, r_2, r_3)$:
$SL(4)$: $r_2 \circ (id \otimes r_3 + r_3 \otimes id) = r_3 \circ (r_2 \otimes id \otimes id - id \otimes r_2 \otimes id + id \otimes id \otimes r_2)$.

We will present an example of an $\mathcal{A}_\infty$-algebra with degree 0 and analyze its properties through specific assignments.

2.15. Example

Let the associative algebras $\mathcal{A}$ be an $\mathcal{A}_\infty$-algebras focused at degree 0 through all multiplications $r_n = 0$ for $n \neq 2$. As a result, the associative algebras make a subclass of $\mathcal{A}_\infty$-algebras with the form $(\mathcal{A}, r_2)$.

We will define morphisms between $\mathcal{A}_\infty$-algebras and describe how to handle these morphisms' necessary conditions and properties.

2.16. Definition [19]

Suppose that $\mathcal{A}$ and $\mathcal{B}$ are both $\mathcal{A}_\infty$-algebras, then the family of M-linear graded maps is the morphism $\hbar: \mathcal{A} \to \mathcal{B}$ of $\mathcal{A}_\infty$-algebras such that:
$$\hbar_n: \mathcal{A}^{\otimes n} \to \mathcal{B} \qquad n \geq 1$$
of degree $(1-n)$, where the subsequent identities apply for each $n \geq 1$:
$$\sum_{\substack{m+s+t=1 \\ m,t\geq 0 \\ s\geq 1}} (-1)^{m+st} \hbar_{m+1+t} \circ \left(id^{\otimes m} \otimes r_s \otimes id^{\otimes t}\right)$$
$$= \sum_{j=1}^{n} \sum_{i_1+\cdots+i_j=n} (-1)^{q} r'_j \left(\hbar_{i1} \otimes \hbar_{i2} \otimes \ldots \otimes \hbar_{ij}\right), \qquad (ML(n))$$
where:
$q = (i_{j-1} - 1) + 2(i_{j-2} - 1) + \cdots + (j-2)(i_2 - 1) + (j-1)(i_1 - 1)$.

Assuming $\mathcal{A}$ and $\mathcal{B}$ are both unital $\mathcal{A}_\infty$-algebras of stringent units $1_{\mathcal{A}}$ and $1_{\mathcal{B}}$ respectively, so $\hbar$ additionally needs to fulfill the following unital morphism requirements:

1) As required for ring morphisms, $\hbar_1(1_{\mathcal{A}}) = 1_{\mathcal{B}}$.
2) If $a_i = 1_{\mathcal{A}}$, then $\hbar_n(a_1 \otimes \ldots \otimes a_n) = 0$ for all $n \geq 2$, $i \in \{1, \ldots, n\}$.
3) A stringent morphism over $\mathcal{A}_\infty$-algebras is $\hbar: \mathcal{A} \to \mathcal{B}$ when $\hbar_n = 0$ for all $n \geq 2$.
4) When $\hbar: \mathcal{A} \to \mathcal{B}$ is a stringent morphism, the morphism identification $ML(n)$ changes to
$\hbar_1 r_n = r_n(\hbar_1 \otimes \ldots \otimes \hbar_1)$.

As a result, ring homeomorphisms and stringent morphisms of $\mathcal{A}_\infty$-algebras tend to be identical.

5) We state that $\hbar$ is a stringent isomorphism if $\hbar_1$ is a vector spaces' isomorphism and $\hbar$ is a stringent morphism. It proves to be even more significant to take into consideration the homology of the $\mathcal{A}_\infty$-algebras, similar to how it is conducted via chain complexes.

Here, we will define strict morphisms and how to handle them within $\mathcal{A}_\infty$-algebras, including the use of these definitions in analysis.

2.17. Definition [19]

The morphism $\hbar$ is considered stringent if $\hbar_i = 0$ for any $i \neq 1$. The stringent morphism $\hbar$ that makes $\hbar_1$ the identity of $\mathcal{A}$ is the identity morphism. While $\hbar: \mathcal{A} \to \mathcal{B}$ is a stringent morphism, the identity $ML(n)$ becomes:
$$\hbar_1 r_n = r_n(\hbar_1 \otimes \cdots \otimes \hbar_1).$$

In classical ring theory, homeomorphisms are similar to strict morphisms.

Morphism $\hbar$ called a strict isomorphism if it is $\hbar$ strict and $\hbar_1$is an isomorphism of vector spaces. In this instance, the inverse morphism of $\hbar$ is denoted by $\hbar_1^{-1}: \mathcal{B} \to \mathcal{A}$.

We will explain how to classify $\mathcal{A}_\infty$-algebras through equivalence classes and how to handle different models of algebras in the following.

2.18. Definition [20]

Assume that $\mathcal{A}$ and $\mathcal{B}$ are $\mathcal{A}_\infty$-algebras and that $\hbar: \mathcal{A} \to \mathcal{B}$ is a $\mathcal{A}_\infty$-morphism. If $\hbar_1$ represents a quasi-isomorphism of complexes, then we can argue that $\hbar$ is a quasi-isomorphism as well. It follows that the induced map $\mathcal{H}(\hbar_1): \mathcal{H}(\mathcal{A}_\bullet) \to \mathcal{H}(\mathcal{B}_\bullet)$ is isomorphic. Here, we use the notation $\mathcal{A} \simeq \mathcal{B}$.
We will classify two $\mathcal{A}_\infty$-algebras as belonging to the same $\mathcal{A}_\infty$-algebras if they are quasi-isomorphic. In other words, quasi-isomorphism classes of $\mathcal{A}_\infty$-algebras will be taken into consideration. A model of $\mathcal{A}$ is a representation of a class of $\mathcal{A}_\infty$-algebras that are quasi-isomorphic to $\mathcal{A}$ and satisfy some helpful properties. As an illustration, the term "minimal model of $\mathcal{A}$" refers to a representative $\mathcal{A}_\infty$-algebras containing a zero r_1. As a result, we can examine $\mathcal{A}_\infty$-algebras by taking into consideration alternative models of $\mathcal{A}$.

Now, we will show that $Ext^*_{\mathcal{A}}(M, M)$ is an $\mathcal{A}_\infty$-algebra and how to use different solutions in this context.

2.19. Theorem [21]

Assuming $\mathcal{A}$ is the algebra over M, then $Ext^*_{\mathcal{A}}(M, M)$ is an $\mathcal{A}_\infty$-algebras.

To demonstrate this, we take the projective resolution $\mathcal{P}$ of ${}_{\mathcal{A}}M$. H Consequently, the differential graded algebra with homology determined by the Yoneda algebra $Ext^*_{\mathcal{A}}(M, M)$ is the morphism complex $\mathcal{B} = Hom_{\mathcal{A}}(\mathcal{P}, \mathcal{P})$.

We will provide an illustrative example of applications of $\mathcal{A}_\infty$-algebras and how to use them in specific cases of tensor products and maps.

2.20. Example

The map r_2 is produced by multiplying R and the mappings $r_n = 0$ for all $n \neq 2$, The graded space $\mathcal{A} = R[\mathfrak{z}]/(\mathcal{E}^2)$ has a trivial $\mathcal{A}_\infty$-structures when R is an extraordinary algebra for $N \geq 1$ and $\mathfrak{z}$ is an undetermined of degree $2 - N$. We define the distorted multiplication and the linear map $\hbar: R^{\otimes N} \to R$ as follows:

$$r'_n = \begin{cases} r_n & n \neq N \\ r_N + \mathcal{E}\hbar & n = N \end{cases}.$$

The given r'_n thus becomes an $\mathcal{A}_\infty$-algebras if and only if $\hbar$ is a simplicial co-cycle over R.

Next, we will define $\mathcal{A}_\infty$-algebras using left models and how to handle fundamental properties of these models.

2.21. Definition [11]

Consider $\mathcal{A}$ as an $\mathcal{A}_\infty$-algebras. The $\mathbb{Z}$-graded vector space R is defined as the left $\mathcal{A}_\infty$-module of $\mathcal{A}$, equipped with maps:

$r_n^R: \mathcal{A}^{\otimes n-1} \otimes R \to R,\ n \geq 1$

of degree $(2 - n)$, fulfilling the identical Stasheff identities $SL(n)$:

$$\sum (-1)^{m+st} r_{m+1+t}(id^{\otimes m} \otimes r_s \otimes id^{\otimes t}) = 0,$$

as in the definition of $\mathcal{A}_\infty$-algebras.

We will describe how to define morphisms between left models of algebras and how to construct these morphisms to meet specified conditions.

2.22. Definition [8]

Assume the morphism of the left $\mathcal{A}_\infty$-modules $\hbar: R \to S$ is defined as the family of the graded maps:
$\hbar_n: \mathcal{A}^{\otimes n-1} \otimes R \to S$
of degree $(1-n)$ for all $n \geq 1$. When $\hbar_1 = id_R$ and $\hbar_i = 0$ for all $i \geq 2$, the identity morphism $\hbar: R \to R$ is produced. The composition of two morphisms $\hbar: R \to S$ and $g: L \to R$ is given by:
$(\hbar \circ g)_n = \sum \hbar_{1+w} \circ (id^{\otimes w} \otimes g_v)$, (4)
where the sum is taken over all decompositions $n = v + w$.

We now classify nested categories and apply functorial transformations between $\mathcal{A}_\infty$-algebras.

2.23. Theorem [8]

Assume that $\hbar: \mathcal{A} \to \mathcal{B}$ is a quasi-isomorphism of $\mathcal{A}_\infty$-algebras. Then, the equivalent functor for triangulated categories that maps $\mathcal{B}$ to $\mathcal{A}$ is:

$\hbar_*: \mathcal{A}_\infty(\mathcal{B}) \to \mathcal{A}_\infty(\mathcal{A})$.

This theorem shows how to establish correspondences between nested categories of $\mathcal{A}_\infty$-algebras and how to use these correspondences in practical applications.

2.24. Theorem [22]

Let $(\mathcal{A}, d_{\mathcal{A}})$ be a differential graded algebra, and let $\mathfrak{C}_{dg}(\mathcal{A})$ e the category of differential graded modules with morphisms between these modules, such that $\mathcal{A}_{dg}(\mathcal{A})$ is the category. Then, the equivalence for the triangulated categories: $\mathcal{A}_{dg}(\mathcal{A}) \to \mathcal{A}_\infty(\mathcal{A})$ is produced by the inclusion functor $\mathfrak{C}_{dg}(\mathcal{A}) \to \mathfrak{C}_\infty(\mathcal{A})$.

Next we define $\mathcal{A}_\infty$-algebras using graded algebra properties and examine how to utilize these definitions in analysis.

2.25. Definition [14]

Assuming $\mathcal{R}$ is a commutative ring and $\mathcal{A} = \sum_{n\in\mathbb{Z}} \mathcal{A}^n$ is a graded $\mathcal{R}$-module. Therefore, the collection of multiplication maps $r_n: \mathcal{A}^{\otimes n} \to \mathcal{A}$ of degree $(2-n)$ is an $\mathcal{A}_\infty$-structures on $\mathcal{A}$, such that:

$$\sum_{m+s+t=n} (-1)^{ms+t} r_{m+1+t}\left(1^{\otimes m} \otimes r_s \otimes 1^{\otimes t}\right) = 0,$$

for all n.

Now, we discuss constructing $\mathcal{A}_\infty$-algebras using homology and applying this structure in various contexts.

2.26. Theorem [23]

Assume $(\mathcal{C}, \Omega, \eta)$ is a differential graded algebra, and $\mathcal{H}_*(\mathcal{C})$ is projective over M in each degree. Then there exists an $\mathcal{A}_\infty$-structure r on $\mathcal{H}_*(\mathcal{C})$ such that:

(i) $r_1 = 0$ (minimal case),
(ii) r_2 is induced by η,
(iii) $(H_*(\mathcal{C}), r)$ is quasi-isomorphic to $(\mathcal{C}, \Omega, \eta)$.

Next, we introduce the Maurer-Cartan equation, which allows us to extend $\mathcal{A}_\infty$-algebras and analyze different applications.

2.27. Definition [24]

Let $\mathcal{H}$ be an $\mathcal{A}_\infty$-algebra, and let $a \in \mathcal{H}_1$, we say a satisfies the Maurer-Cartan equation if:

$$\sum_{n\in\mathbb{N}} \pm r_n(a_{\otimes n}) = 0.$$

The twisted $\mathcal{A}_\infty$-algebras $\mathcal{H}_a$ is then defined by:

$$r_n^a(i_1,\cdots,i_n) = \sum_{\ell_1+\cdots+\ell_{n+1}\in\mathbb{N}_0} \pm r_{n+\ell}\big(a_{\otimes\ell_1}, i_1, a_{\otimes\ell_2}, i_2, \cdots, i_n, a_{\otimes\ell_{n+1}}\big),$$

where $\ell = \ell_1 + \cdots + \ell_{n+1}$.

These formulas also apply to $\mathcal{A}_\infty$-bimodules. Hence, if $\mathcal{N}$ is an $\mathcal{A}_\infty$-bimodules over $\mathcal{H}$ then, $\mathcal{N}_a$ is an $\mathcal{A}_\infty$-bimodules over $\mathcal{H}_a$.

Now, we define simple homology for $\mathcal{A}_\infty$-algebras using complexes and their sequences.

2.28. Definition [25]

Let a space $\mathcal{X}$ be an $\mathcal{A}_\infty$-algebra, and let $(\mathcal{X}_*, d_*) = \{\mathcal{X}_n, d_n\}$ represent a chain complex such that:

$$\dots \longrightarrow \mathcal{X}_{n+1} \xrightarrow{d_{n+1}} \mathcal{X}_n \xrightarrow{d_n} \mathcal{X}_{n-1} \xrightarrow{d_{n-1}} \dots \xrightarrow{d_1} \mathcal{X}_0 \xrightarrow{d_0} 0$$

Then, the n^{th} homology of an $\mathcal{A}_\infty$-algebra $\mathcal{X}$ is defined as:

$$\mathcal{H}_n(\mathcal{X}) = \frac{ker(d_n)}{Im(d_{n+1})}. \quad (5)$$

Since $Im\, d_{n+1} \subset ker\, d_n$, and $\mathcal{H}_n(\mathcal{X}) = \frac{n\text{−cycles}}{n\text{−boundaries}}$.

Finally, we define $\mathcal{A}_\infty$-algebras using corrections for algebra models and explain how to handle derived structures.

2.29. Definition [22]

Let $\mathcal{B}$ be a unital associative algebra, R a right $\mathcal{B}$-module, and $\mathcal{P} \to R$ the projective resolution. Consider $\mathcal{A} = Hom_{\mathcal{B}}(\mathcal{P},\mathcal{P})$ represent the differential graded endomorphism algebras of $\mathcal{P}$, with its n^{th} component consisting of the morphisms of graded items of degree n and its differential being the super commutator corresponding to the differential of $\mathcal{P}$.

Since $\mathcal{A}$ is specifically an $\mathcal{A}_\infty$-algebra, it includes a minimal model. The homology $\mathcal{H}_*\mathcal{A}$, as an algebra of r_2, is now isomorphic to the Yoneda algebra $Ext_{\mathcal{B}}^*(R,R)$.

Next, we explain simple homology for $\mathcal{A}_\infty$-algebras and how to use these definitions in various applications.

2.30. Definition [26]

Assume that $(\mathcal{A} = \bigoplus_{j\in\mathbb{Z}} \mathcal{A}_j, (\eta_n)_{n\in\mathbb{N}})$ is an $\mathcal{A}_\infty$-algebras over M and let $(\mathcal{S} = \bigoplus_{j\in\mathbb{Z}} \mathcal{S}_j, (\mu_{m,s}^{\mathcal{S}})_{m,s\in\mathbb{N}_0})$ be an $\mathcal{A}_\infty$-bimodules over $\mathcal{A}$. We further denote the index of $r \in \mathcal{S}$ by $\mu(r) := \mu_{\mathcal{S}}(r)$. Consider the graded M-bimodule:

$$\mathcal{HH}_*(\mathcal{A},\mathcal{S}) := \bigoplus_{n=0}^{\infty} \mathcal{S} \otimes \mathcal{A}^{\otimes n},$$

with grading on $\mathcal{HH}_*(\mathcal{A},\mathcal{S})$ is given by:

$$\mathcal{HH}_j(\mathcal{A},\mathcal{S}) = \bigoplus_{n\in\mathbb{N}_0} \bigoplus_{j=n-j_0-j_1-\cdots-j_n} \mathcal{S}_{j_0} \otimes \mathcal{A}_{j_1} \otimes \dots \otimes \mathcal{A}_{j_n}. \quad (6)$$

which corresponds to the simplicial homology of $\mathcal{A}_\infty$-algebras.

After that, we discuss how to handle homology degrees in graded spaces and explain how to determine these degrees using advanced definitions.

2.31. Definition [27]

For $a \in \mathcal{HH}_*(\mathcal{A},\mathcal{S})$ we write $deg(a) = j$ if and only if $a \in \mathcal{HH}_j(\mathcal{A},\mathcal{S})$ and call it the degree of a. Note that for all $s \in \mathcal{S}, n \in \mathbb{N}_0$ and $a_1, a_2, \dots, a_n \in \mathcal{A}$ the degree of $s \otimes a_1 \otimes \dots \otimes a_n$ is explicitly given by:

$$\deg(s \otimes a_1 \otimes \dots \otimes a_n) = n - \eta(s) - \sum_{j=1}^{n} \eta(a_j) = -\eta(s) - \sum_{j=1}^{n} \|a_j\|.$$

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Remark: The degree on $\mathcal{HH}_*(\mathcal{A},\mathcal{S})$ is best understood in terms of the shifted $\mathcal{A}_\infty$-algebras $\mathcal{A}$ [1]. We may identify $\mathcal{HH}_*(\mathcal{A},\mathcal{S})$ as a group, with $\bigoplus_{n=0}^{\infty} \mathcal{S}\otimes\mathcal{A}^{\otimes n}$. The degree on $\mathcal{HH}_*(\mathcal{A},\mathcal{S})$ then coincides with the usual product degree of this tensor algebras.

Next, we describe how to deal with the relative homology of algebras and how to use suitable models for analysis in this context.

2.32. Definition [28]

There is a map for the relative homology of $\mathcal{A}$ modulo $\mathcal{I}$ if $\mathcal{A}$ is an $\mathcal{A}_\infty$-algebras and $\mathcal{I}$ is an ideal, where $\mathcal{A}\longrightarrow\mathcal{A}/\mathcal{I}$ is $\mathcal{A}_\infty$-split, such that:

$$\mathfrak{z}: \mathcal{HH}_n(\mathcal{I})\longrightarrow\mathcal{HH}_n(\mathcal{A}/\mathcal{I}),$$

If this map is an isomorphism, it is claimed that the ideal $\mathcal{I}$ is the excision of simplicial homology. This leads to the following exact sequence:

$$\ldots\longrightarrow\mathcal{HH}_n(\mathcal{I})\longrightarrow\mathcal{HH}_n(\mathcal{A})\longrightarrow\mathcal{HH}_n(\mathcal{A}/\mathcal{I})\longrightarrow\mathcal{HH}_{n-1}(\mathcal{I})\longrightarrow\mathcal{HH}_{n-1}(\mathcal{A})\longrightarrow\ldots$$

Finally, the exact sequence for $\mathcal{A}_\infty$-algebras with ideal $\mathcal{I}$ includes a boundary map δ, which is clearly defined in terms of relative cycles.

2.33. Proposition [29]

Given $\mathcal{A}$ be $\mathcal{A}_\infty$-algebras and $\mathcal{I}$-ideal since $\mathcal{I}\subset\mathcal{A}$, we have the following exact sequence:

$$\mathcal{HH}_n(\mathcal{I}) \xrightarrow{i_*} \mathcal{HH}_n(\mathcal{A})$$

$\delta(-1)$ $\qquad$ j_*

$$\mathcal{HH}_n(\mathcal{A}/\mathcal{I})$$

Actually, in this application of relative homology, the boundary map δ contains an obvious description: The $(n-1)$-homology class provided by $[\delta_\varphi]\in\mathcal{H}_{n-1}(\mathcal{A})$ is $\delta[\varphi]$ if $\varphi\in\mathcal{C}_n(\mathcal{Y},\mathcal{A})$ denotes a relative cycle.

3. Main Result

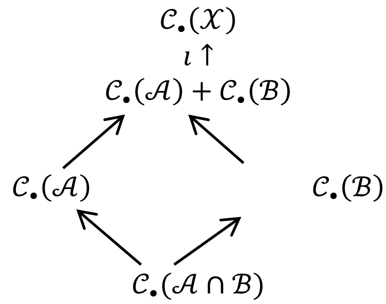
This text explores essential topics in algebraic topology and homology related to $\mathcal{A}_\infty$-algebras. It covers how isomorphisms between homology groups are preserved under specific conditions, defines simplicial and bar homology with module coefficients, and discusses $\mathcal{H}$-unitarity. The discussion also includes the relationships between different homological constructs and the conditions required for quasi-isomorphisms, providing a comprehensive view of these concepts and their interconnections.

The frist theorem discusses the excision theorem for $\mathcal{A}_\infty$-algebras, demonstrating how isomorphisms between homology groups persist under specific conditions and the inclusion maps. It includes proof strategies involving chain complexes and homotopy equivalence.

3.1. Theorem (Excision Theory)

Suppose that $\mathcal{E}$ is a subset of $\mathcal{A}_\infty$-algebras such that $\mathcal{E}\subset\mathcal{A}\subset\mathcal{X}$. Then, for all n the isomorphisms $\mathcal{H}_n(\mathcal{X}\backslash\mathcal{E},\mathcal{A}\backslash\mathcal{E})\longrightarrow\mathcal{H}_n(\mathcal{X},\mathcal{A})$ that given by the inclusion $(\mathcal{X}\backslash\mathcal{E},\mathcal{A}\backslash\mathcal{E})\hookrightarrow(\mathcal{X},\mathcal{A})$.
If $\mathcal{X}$ is covered by the interiors of the spaces $\mathcal{A},\mathcal{B}$ such that $\mathcal{A},\mathcal{B}\subset\mathcal{X}$, then the inclusion $(\mathcal{B},\mathcal{A}\cap\mathcal{B})\hookrightarrow(\mathcal{X},\mathcal{A})$ is the equivalent statement that persuades the isomorphisms $\mathcal{H}_n(\mathcal{B},\mathcal{A}\cap\mathcal{B})\to\mathcal{H}_n(\mathcal{X},\mathcal{A})$ for all n, where the space $\mathcal{B}$ is given by $\mathcal{B}=\mathcal{X}\backslash\mathcal{E}$.

Proof: By using [30], [31] and consider $\mathcal{X}$ as the combination of $\mathcal{A}$ and $\mathcal{B}$ through interiors covering $\mathcal{X}$. Next, there are maps of natural inclusion.



We would obtain: $\mathcal{C}_\bullet(\mathcal{X})/\mathcal{C}_\bullet(\mathcal{A}) = \mathcal{C}_\bullet(\mathcal{B})/\mathcal{C}_\bullet(\mathcal{A} \cap \mathcal{B})$,

where the map ι remains an isomorphism, which leads to the desired result. However, there are terrible simplices that can have non-empty intersections with $(\mathcal{A} - \mathcal{A} \cap \mathcal{B})$ and $(\mathcal{B} - \mathcal{A} \cap \mathcal{B})$; therefore, the fact that the map ι is not an isomorphism is an issue. Assisted by the chain map $\xi: \mathcal{C}_\bullet(\mathcal{X}) \longrightarrow \mathcal{C}_\bullet(\mathcal{A}) + \mathcal{C}_\bullet(\mathcal{B})$, we would desire to demonstrate how to split up terrible simplices into little good ones without changing the homology.

We demonstrate that $\mathcal{C}_\bullet(\mathcal{A}) + \mathcal{C}_\bullet(\mathcal{B})$ be the distorted retracting of $\mathcal{C}_\bullet(\mathcal{X})$, indicating that $\xi \circ \iota = Id$ and $\iota \circ \xi = d\mathfrak{D} + \mathfrak{D}d$ for particular chain homotopy $\mathfrak{D}$.

In order to retain the sub-complexes $\mathcal{C}_\bullet(\mathcal{A})$ and $\mathcal{C}_\bullet(\mathcal{B})$, we choose $\mathfrak{D}$, indicating that we achieve the equivalence: $\mathcal{C}_\bullet(\mathcal{X})/\mathcal{C}_\bullet(\mathcal{A}) \longrightarrow \mathcal{C}_\bullet(\mathcal{B})/\mathcal{C}_\bullet(\mathcal{A} \cap \mathcal{B})$ as chain homotopy equivalence.

Next, we define simplicial homology isomorphisms in the context of $\mathcal{A}_\infty$-algebras, emphasizing the behavior of these isomorphisms under inclusions of subspaces.

3.2. Definition

For the space $\mathcal{X}$ and $\mathcal{E} \subset \mathcal{A}_\infty$-algebras where $\mathcal{E} \subset \mathcal{A} \subset \mathcal{X}$, the simplicial homology isomorphisms induced by the inclusion $(\mathcal{X} \backslash \mathcal{E}, \mathcal{A} \backslash \mathcal{E}) \hookrightarrow (\mathcal{X}, \mathcal{A})$ for all n is:

$\mathcal{HH}_n(\mathcal{X} \backslash \mathcal{E}, \mathcal{A} \backslash \mathcal{E}) \longrightarrow \mathcal{HH}_n(\mathcal{X}, \mathcal{A})$.

By setting the space $\mathcal{B} = \mathcal{X} \backslash \mathcal{E}$, let $\mathcal{X}$ covered by the interiors of the spaces $\mathcal{A}, \mathcal{B}$ for $\mathcal{A}, \mathcal{B} \subset \mathcal{X}$, then the equivalent statement is that the following isomorphisms persuaded by the inclusion $(\mathcal{B}, \mathcal{A} \cap \mathcal{B}) \hookrightarrow (\mathcal{X}, \mathcal{A})$:

$$\mathcal{HH}_n(\mathcal{B}, \mathcal{A} \cap \mathcal{B}) \to \mathcal{HH}_n(\mathcal{X}, \mathcal{A}) \quad \forall n. \tag{7}$$

Now, we introduce the bar homology of $\mathcal{A}_\infty$-algebras with coefficients in a module detailing the associated boundary maps.

3.3. Definition

Assuming that $\mathcal{J}$ is an $\mathcal{A}_\infty$-algebras, which is not necessarily unitary and make $\mathcal{R}$ a right $\mathcal{J}$-module. Then, the complexes' homology $\mathcal{H}B'_*(\mathcal{J}, \mathcal{R})$ is the bar homology of $\mathcal{J}$ via coefficients in $\mathcal{R}$:

$$(\mathcal{R} \otimes \mathcal{J}^{\otimes *}, \rho'_*) := \mathcal{R} \xleftarrow{\rho'_1} \mathcal{R} \otimes \mathcal{J} \xleftarrow{\rho'_2} \mathcal{R} \otimes \mathcal{J} \otimes \mathcal{J} \xleftarrow{\rho'_3} \mathcal{R} \otimes \mathcal{J} \otimes \mathcal{J} \otimes \mathcal{J} \xleftarrow{\rho'_4} \dots$$

such that a tensor product obtained over $\mathcal{A}_\infty$-algebras and a boundary map is provided by:

$\rho'_n(a_0 \otimes \dots \otimes a_n) = \sum_{i=0}^{n-1} (-1)^i a_0 \otimes \dots \otimes a_i a_{i+1} \otimes \dots \otimes a_n$.

The following definition defines the simplicial homology of complexes and explores the boundary maps involved, illustrating their connection to the bar homology definitions.

3.4. Definition

The homology $\mathcal{HH}_*(\mathcal{J}, \mathcal{R})$ of complexes represents the simplicial homology of $\mathcal{J}$ via coefficients in $\mathcal{R}$:

$$(\mathcal{R} \otimes \mathcal{J}^{\otimes *}, \rho_*) := \mathcal{R} \xleftarrow{\rho_1} \mathcal{R} \otimes \mathcal{J} \xleftarrow{\rho_2} \mathcal{R} \otimes \mathcal{J} \otimes \mathcal{J} \xleftarrow{\rho_3} \mathcal{R} \otimes \mathcal{J} \otimes \mathcal{J} \otimes \mathcal{J} \xleftarrow{\rho_4} \dots,$$

with the boundary map provided by:

$$\rho_n(a_0 \otimes \dots \otimes a_n) = \rho'_n(a_0 \otimes \dots \otimes a_n) + (-1)^n a_n a_0 \otimes a_1 \otimes \dots \otimes a_{n-1}.$$

We present a corollary relating to H-homology and bar homology, establishing the relationships between different homological constructs for $\mathcal{A}_\infty$-algebras and discussing exact sequences that arise.

3.5. Corollary

The $\mathcal{A}_\infty$-algebras that are produced from $\mathcal{I}$ by adding the value of unity to $\mathcal{I}$ be denoted by $\tilde{\mathcal{I}} = k \times \mathcal{I}$. Then the H-homology $H_*(\mathcal{I})$ is defined by $\mathcal{H}_*(\mathcal{I}) := \mathcal{H}_*(\mathcal{I}, \mathcal{I})$, the bar homology $\mathcal{H}B_*(\mathcal{I})$ is defined by $\mathcal{H}B_*(\mathcal{I}) := \mathcal{H}B'_*(\mathcal{I}, \mathcal{I})$, and $\mathcal{HH}_*(\mathcal{I}) := \bar{\mathcal{H}}_*(\mathcal{I}, \tilde{\mathcal{I}})$ defines the simplicial homology of $\mathcal{I}$, where $\bar{\mathcal{H}}_n(\mathcal{I}, \tilde{\mathcal{I}}) = \mathcal{H}_n(\mathcal{I}, \tilde{\mathcal{I}})$, $\forall\, n > 0$ and $\bar{\mathcal{H}}_0(\mathcal{I}, \tilde{\mathcal{I}}) = \mathcal{H}_0(\mathcal{I}, \tilde{\mathcal{I}})/k$. Let the homology $\mathcal{HH}_*(\mathcal{I})$ be the double complex' homology such that:

$$\mathcal{CC}(\mathcal{I})^{|2|} := \left(\mathcal{I} \otimes \mathcal{I}^{\otimes *}, \rho_*\right) \xleftarrow{1-t} \left(\mathcal{I} \otimes \mathcal{I}^{\otimes *}, -\rho'_*\right). \qquad (8)$$

Consequently, the exact sequence exists as follows:

$$\dots \longleftarrow H_{n-1}(\mathcal{I}) \longleftarrow \mathcal{H}B_{n-1}(\mathcal{I}) \longleftarrow \mathcal{HH}_n(\mathcal{I}) \longleftarrow H_n(\mathcal{I}) \longleftarrow \mathcal{H}B_n(\mathcal{I}) \longleftarrow \mathcal{HH}_{n+1}(\mathcal{I}) \longleftarrow \dots.$$

The following definition covers the concept of $\mathcal{H}$-unitarity in $\mathcal{A}_\infty$-algebras, focusing on the conditions under which a module $\mathcal{R}$ is considered $\mathcal{H}$-unitary.

3.6. Definition

Suppose that $\mathcal{R}$ is an $\mathcal{I}$-bimodule since $\mathcal{I}$ is an $\mathcal{A}_\infty$-algebras. If every $\mathcal{A}_\infty$-modules $\mathcal{G}$ has an exact complex $\left(\mathcal{R} \otimes \mathcal{I}^{\otimes *}, \rho_*\right) \otimes \mathcal{G}$, we can deduce that $\mathcal{R}$ is $\mathcal{H}$-unitary.

One states that $\mathcal{I}$ is $\mathcal{H}$-unital when $\mathcal{R} = \mathcal{I}$, such $\mathcal{R}$ is the left $\mathcal{I}$-module. It follows logically that $\mathcal{R} \otimes \mathcal{I}$ is $\mathcal{H}$-unitary, if $\mathcal{I}$ is $\mathcal{H}$-unital.

In the following, we examine a theorem on quasi-isomorphisms between complexes in the context of $\mathcal{A}_\infty$-algebras and bimodules, proving the results under specific assumptions about $\mathcal{H}$-unitarity.

3.7. Theorem

Assume that the extension of $\mathcal{A}_\infty$-algebras is given by $0 \longrightarrow \mathcal{I} \longrightarrow \mathcal{A} \longrightarrow \mathcal{B} \longrightarrow 0$, and defines $\mathcal{G}$ to be a $\mathcal{A}_\infty$-modules and $\mathcal{R}$ to be an $\mathcal{A}$-bimodule. Then we can say that the following canonical inclusions:

$$i: \left(\mathcal{R} \otimes \mathcal{I}^{\otimes *}, \rho_*\right) \otimes \mathcal{G} \hookrightarrow \left(\mathcal{R} \otimes \mathcal{A}^{\otimes *}, \rho_*\right) \otimes \mathcal{G}, \qquad (9)$$

$$i': \left(\mathcal{R} \otimes \mathcal{I}^{\otimes *}, \rho'_*\right) \otimes \mathcal{G} \hookrightarrow \left(\mathcal{R} \otimes \mathcal{A}^{\otimes *}, \rho'_*\right) \otimes \mathcal{G} \qquad (10)$$

are quasi-isomorphisms when the $\mathcal{I}$-bimodule $\mathcal{R}$ is $\mathcal{H}$-unitary.

Proof: By considering the filtration $F^0 \subseteq F^0 \subseteq \cdots$ of $\left(\mathcal{R} \otimes \mathcal{A}^{\otimes *}, \rho_*\right)$ and using [32], [33], assume that $\mathcal{G}$ is an $\mathcal{A}_\infty$-modules, such that:

$$F^\ell := \mathcal{R} \xleftarrow{\rho_1} \mathcal{R} \otimes \mathcal{A} \xleftarrow{\rho_2} \mathcal{R} \otimes \mathcal{A}^{\otimes 2} \xleftarrow{\rho_3} \dots \xleftarrow{\rho_\ell} \mathcal{R} \otimes \mathcal{A}^{\otimes p} \xleftarrow{\rho_{\ell+1}} \mathcal{R} \otimes \mathcal{I} \otimes \mathcal{A}^{\otimes \ell} \xleftarrow{\rho_{\ell+2}} \mathcal{R} \otimes \mathcal{I}^{\otimes 2} \otimes \mathcal{A}^{\otimes \ell} \xleftarrow{\rho_{\ell+3}} \dots$$

For all $\ell \geq 0$, we have:

$$\left(F^{\ell+1} \otimes \frac{\mathcal{G}}{F^\ell} \otimes \mathcal{G}\right)_* = \left(\mathcal{R} \otimes \mathcal{I}^{\otimes\, *-\ell-1}, \rho'_*\right) \otimes \mathcal{B} \otimes \mathcal{A}^{\otimes \ell} \otimes \mathcal{G}, \qquad (11)$$

According to theory, this is exact. Taking into consideration the homology of long exact sequence for all $n \geq 0$ connected to

$$0 \longrightarrow F^n \otimes \mathcal{G} \longrightarrow F^{n+1} \otimes \mathcal{G} \longrightarrow \frac{F^{n+1} \otimes \mathcal{G}}{F^n \otimes \mathcal{G}} \longrightarrow 0 \qquad (12)$$

As can be seen, the canonical map $F^0 \longrightarrow F^\ell$ represents quasi-isomorphism for every ℓ, i is also a quasi-isomorphism consequently.

In a similar demonstration, the same applies to i'.

Remark: Note that the theorem (3.7) given above may also be proved in the case when $\mathcal{I}$ is a right ideal of $\mathcal{A}$ instead of a two-sided ideal.

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The following corollary provides further insight into quasi-isomorphisms in the context of extensions of $\mathcal{A}_\infty$-algebras, focusing on modules and the conditions for $\mathcal{H}$-unitarity.

3.8. Corollary

Suppose that $0 \longrightarrow \mathcal{J} \longrightarrow \mathcal{A} \longrightarrow \mathcal{B} \longrightarrow 0$ is an extension of the $\mathcal{A}_\infty$-algebras where $\mathcal{J} \subset \mathcal{A} \subset \mathcal{B}$ and $\mathcal{G}$ is a k-module and use [34]. The canonical arrows:

$$\pi: \left(\mathcal{B} \otimes \mathcal{A}^{\otimes *}, \rho_*\right) \otimes \mathcal{G} \longrightarrow \left(\mathcal{B} \otimes \mathcal{B}^{\otimes *}, \rho_*\right) \otimes \mathcal{G},$$
$$\pi': \left(\mathcal{B} \otimes \mathcal{A}^{\otimes *}, {\rho'}_*\right) \otimes \mathcal{G} \longrightarrow \left(\mathcal{B} \otimes \mathcal{B}^{\otimes *}, {\rho'}_*\right) \otimes \mathcal{G},$$

are quasi-isomorphisms when $\mathcal{J}$ is an $\mathcal{H}$-unital.

Proof: We must take into consideration for all $\ell \geq 0$, the quotient complex $\tilde{F}^\ell$ for $(\mathcal{B} \otimes \mathcal{A}^{\otimes *}, \rho_*)$ provided by:

$$\tilde{F}^\ell \coloneqq \mathcal{B} \xleftarrow{\rho_1} \mathcal{B} \otimes \mathcal{B} \xleftarrow{\rho_2} \mathcal{B} \otimes \mathcal{B}^{\otimes 2} \xleftarrow{\rho_3} \dots \xleftarrow{\rho_\ell} \mathcal{B} \otimes \mathcal{B}^{\otimes \ell} \xleftarrow{\rho_{\ell+1}} \mathcal{B} \otimes \mathcal{B}^{\otimes \ell} \otimes \mathcal{A} \xleftarrow{\rho_{\ell+2}} \mathcal{B} \otimes \mathcal{B}^{\otimes \ell} \otimes \mathcal{A}^{\otimes 2} \xleftarrow{\rho_{\ell+3}} \dots .$$

to demonstrate whether π is a quasi-isomorphism, consider that the canonical projections

$\pi^\ell: \tilde{F}^\ell \otimes \mathcal{G} \longrightarrow \tilde{F}^{\ell+1} \otimes \mathcal{G}$.

Given that $\mathcal{B}(\ell) = \mathcal{B} \otimes \mathcal{B}^{\otimes \ell} \otimes \mathcal{J}$, the straightforward calculation reveals that:

$Ker(\pi^\ell) = (\mathcal{B}(\ell) \otimes \mathcal{A}^{\otimes * - \ell - 1}, \rho_*) \otimes \mathcal{G}$.

Therefore, according to Theorem (3.7), $Ker(\pi^\ell)$ is quasi-isomorphic to:

$$\left(\mathcal{B}(\ell) \otimes \mathcal{J}^{\otimes * - \ell - 1}, \rho_*\right) \otimes \mathcal{G} = (\mathcal{B}(\ell) \otimes \mathcal{J}^{\otimes * - \ell - 1}, \rho'_*) \otimes \mathcal{G}, \qquad (13)$$

which is exact by assumption. For π', a similar proof applies.

Now, we conclude with a theorem establishing the equivalence of various conditions related to $\mathcal{H}$-unitarity, excision, and homology requirements for $\mathcal{A}_\infty$-algebras, providing a comprehensive view of their interrelations.

3.9. Theorem

Let the assumption that $\mathcal{J}$ is a $\mathcal{A}_\infty$-algebras, and then the next propositions are equivalent:

(1) The $\mathcal{A}_\infty$-algebras $\mathcal{J}$ remains $\mathcal{H}$-unital.
(2) The $\mathcal{A}_\infty$-algebras $\mathcal{J}$ fulfills the H-homology excision.
(3) The $\mathcal{A}_\infty$-algebras $\mathcal{J}$ fulfills the excision requirement of bar homology.
(4) The $\mathcal{A}_\infty$-algebras $\mathcal{J}$ fulfills the excision requirement of simplicial homology.

Proof: The proposition (1) is equivalent to (2) when we let $0 \to \mathcal{J} \to \mathcal{A} \to \mathcal{B} \to 0$ be an $\mathcal{A}_\infty$-algebras as a pure extension and $\mathcal{G}$ be a k-module, and the canonical projection given by:

$\pi: (\mathcal{A} \otimes \mathcal{A}^{\otimes *}, \rho_*) \otimes \mathcal{G} \to (\mathcal{B} \otimes \mathcal{B}^{\otimes *}, \rho_*) \otimes \mathcal{G}$.

Suppose that the commutation diagram of short exact sequences is as follows:

$$\begin{array}{ccccccccc} 0 & \to & (\mathcal{J} \otimes \mathcal{A}^{\otimes *}, \rho_*) \otimes \mathcal{G} & \to & (\mathcal{A} \otimes \mathcal{A}^{\otimes *}, \rho_*) \otimes \mathcal{G} & \to & (\mathcal{B} \otimes \mathcal{A}^{\otimes *}, \rho_*) \otimes \mathcal{G} & \to & 0 \\ & & |j & & | = & & |\pi_1 & & \\ 0 & \to & ker(\pi) & \to & (\mathcal{A} \otimes \mathcal{A}^{\otimes *}, \rho_*) \otimes \mathcal{G} & \overset{\pi}{\to} & (\mathcal{B} \otimes \mathcal{A}^{\otimes *}, \rho_*) \otimes \mathcal{G} & \to & 0 \end{array}$$

According to Corollary (3.8), π_1 indicates a quasi-isomorphism. As a result, j is as well. We continue the proof using the Theorem (3.7).

The proposition (1) is equivalent to (3): Comparable to (1) $\Longrightarrow$ (2).

The proposition (2) and (3) is equivalent to (4): The long exact sequence:

$$\cdots \leftarrow H_{n-1}(\mathcal{J}) \leftarrow \mathcal{H}B_{n-1}(\mathcal{J}) \leftarrow \mathcal{H}\mathcal{H}_n(\mathcal{J}) \leftarrow H_n(\mathcal{J}) \leftarrow \mathcal{H}B_n(\mathcal{J}) \leftarrow \mathcal{H}\mathcal{H}_{n+1}(\mathcal{J}) \leftarrow \cdots$$

gives this simple consequence.

The proposition (2) is equivalent to (1): Assume that k-algebra $\mathcal{A} = \mathcal{J} \oplus \mathcal{G}$ with k-module $\mathcal{G}$ in addition to the canonical projection:

$\pi: (\mathcal{A} \otimes \mathcal{A}^{\otimes *}, \rho_*) \to (\mathcal{G} \otimes \mathcal{G}^{\otimes *}, \rho_*)$,

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where the product is provided by $(u,v)(u',v') = (uu',0)$. Given that $ker\,(\pi)$ is a direct summation of the complex $\mathcal{G} \otimes (\mathcal{I} \otimes \mathcal{I}^{\otimes\, *-1}, \rho'_*) \oplus (\mathcal{I} \otimes \mathcal{I}^{\otimes *}, \rho'_*)$,
$\mathcal{I}$ satisfy the excision to H-homology. In such a case, $\mathcal{G} \otimes (\mathcal{I} \otimes \mathcal{I}^{\otimes\, *-1}, \rho'_*)$ is exact.
The proposition (3) is equivalent to (1): Comparable to (2) $\Longrightarrow$ (1).
The proposition (4) is equivalent to (1): Allow $\mathcal{A}$ and $\mathcal{G}$ are being in (2) $\Longrightarrow$ (1).
Let the canonical projection given by $\bar{\pi}: \mathcal{C}_{**}(\mathcal{A}) \to \mathcal{C}_{**}(\mathcal{G})$ and β be the sub-complex of $ker\,(\pi)$ produced by the components $(a_0 \otimes \cdots \otimes a_n, a'_0 \otimes \cdots \otimes a'_{n-1})$ that include some a_i with some a'_n in $\mathcal{G}$.
So β is exact such that $ker(\bar{\pi}) = \mathcal{C}_{**}(\mathcal{I}) \oplus \beta$ and $\mathcal{I}$ fulfills excision instead of simplicial homology.
Consider the case when $\mathcal{I}$ cannot be $\mathcal{H}$-unital. Suppose $x \in \mathcal{G} \otimes \mathcal{I}^{\otimes n}$ represents a cycle that does not represent a boundary for ρ'_n, it is obvious that $(0, N(x))$ in is a cycle of degree $n+1$ that is not a boundary, which is in direct opposition to the exactness of β.

3.7. Concluding Remarks

The core focus was on $\mathcal{A}_\infty$-algebras, defined by Stasheff identities that ensure homotopy associativity. We examined various examples and their applications, highlighting their versatility. Morphisms between $\mathcal{A}_\infty$-algebras were also analyzed, including strict morphisms and quasi-isomorphisms.

Our study further explored the homology of $\mathcal{A}_\infty$-algebras, specifically simplicial homology, and detailed the relationship between bar homology $\mathcal{H}B_{n-1}(\mathcal{I})$ and simplicial homology $\mathcal{H}\mathcal{H}_n(\mathcal{I})$ as an exact sequence. We demonstrated several quasi-isomorphisms and presented a commutative diagram illustrating these relationships. In summary, the homology theory of $\mathcal{A}_\infty$-algebras extends traditional algebraic concepts into a homotopical framework, offering a deep understanding of their foundational aspects and broader implications in mathematical research.

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