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# Mathematical explorations on the sequence of factoriangular numbers: Extending the results on generalizations

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**Abstract:** A factoriangular number is formed by adding a factorial and a triangular number. If corresponding factorials and triangular numbers are added, the results are $n$-factoriangular numbers. Other factoriangular numbers are called $(n,k)$-factoriangular numbers, $n^{(m)}$-factoriangular numbers, $(n^{(m)},k^{(m)})$-factoriangular numbers, and $(n^{(a)},k^{(b)})$-factoriangular numbers. The main objective of this study is to explore the sequence of $(n^{(m)},k^{(m)})$-factoriangular numbers and the sequence of $(n^{(a)},k^{(b)})$-factoriangular numbers as generalizations of the sequence of $n$-factoriangular numbers. This research is a discipline-based scholarship of discovery that employs an exploratory method involving the scientific approach of experimental mathematics. The mathematical method was used in doing the explorations, focusing on the formulations and proofs of theorems and giving some examples. For the main results, ten theorems were proven and several examples of sequences were provided. The theorems include several formulas for $(n^{(m)},k^{(m)})$-factoriangular numbers, and $(n^{(a)},k^{(b)})$-factoriangular numbers. The proofs for theorems in $(n^{(m)},k^{(m)})$-factoriangular numbers are applicable for similar theorems in $(n^{(a)},k^{(b)})$-factoriangular numbers. Specific sequences of some generalized factoriangular numbers were presented in tables. Entries of numbers in the tables may lead to the formation of triangular arrays of factoriangular numbers that may be further explored by other researchers, especially those mostly interested in recreational mathematics.
***Keywords:*** *Factorial, Factoriangular number, Generalization, Integer sequence, Number theory, Triangular number.*

## 1. Introduction

Mathematics literature provide a long history of research in triangular numbers and factorials. It is commonly argued that triangular numbers were already known to the ancient Greeks, who viewed them with reverence [1] and most especially to the Pythagoreans who discoursed on the number of dots or pebbles that could form geometrical figures, such as a triangle [2]. While the triangular numbers were known to the Pythagoreans of ancient Greece, the factorials were known to the Jains of ancient India and to the Hebrews of ancient Middle East. Although Greek mathematics included combinatorics, there is no direct evidence of ancient Greek study of factorials. It is in about mid-17th to the early 18th century that the factorial function was intensively studied by leading mathematicians of the period including [3-5]. The literature also provides some expositions on triangular numbers [1] early works on factorial function [3] and some early and recent studies on triangular numbers, factorials, and other related numbers [6].

Generalization is an important part of mathematics and it serves as a tool in constructing new knowledge [7]. Mathematical generalization encompasses a claim that some property or techniques holds for a set of mathematical objects or conditions, the scope of which is always larger than the set of individually verified cases [8]. Like any other theorem, a generalization is accepted to be true if and only if it is supported by a valid proof. Generalizations have been applied to a variety of number-theoretic problems. The triangular numbers, the factorials, and many number-theoretic theorems


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related to them have been generalized in several ways in the previous studies. For instance, an expository article discusses some generalized factorials [9]. A more recent article generalizes triangular numbers to arbitrary higher-dimensional spaces [10].

Triangular numbers and factorials are associated in such a way that triangular numbers are the additive analogs of the factorials [11]. The apparent natural connection between these two sequences of numbers contributed to the interest of adding corresponding factorials and triangular numbers to form a new sequence of integers, which is called factoriangular numbers [12]. This relatively new sequence is included in The Online Encyclopedia of Integer Sequences (OEIS) as Entry A101292 [13]. With the introduction of factoriangular numbers, the literature now provides Fibonacci factoriangular numbers [14] Pell factoriangular numbers [15] Lucas factoriangular numbers [16] factoriangular numbers in balancing and Lucas-balancing sequence [17] and multiple factoriangular numbers [18].

The multiple factoriangular number is a generalization of the factoriangular number. Several articles also discuss some other generalizations of factoriangular numbers [19, 20]. In this expository paper, we provide some explorations on further generalizations of factoriangular numbers.

## 2. Methodology

This expository article is a result of discipline-based scholarship of discovery particularly, basic research in number theory. We employ an exploratory method involving the scientific approach of experimental mathematics. Experimental mathematics is the methodology of doing mathematics that includes the use of computations for gaining insight and intuition, discovering new patterns and relationships, using graphical displays to suggest underlying mathematical principles, testing and especially falsifying conjectures, exploring a possible result to see if it is worth formal proof, suggesting approaches for formal proof, replacing lengthy hand derivations with computer-based derivations, and confirming analytically derived results [21, 22]. More particularly, we use the mathematical method [22] presented below:

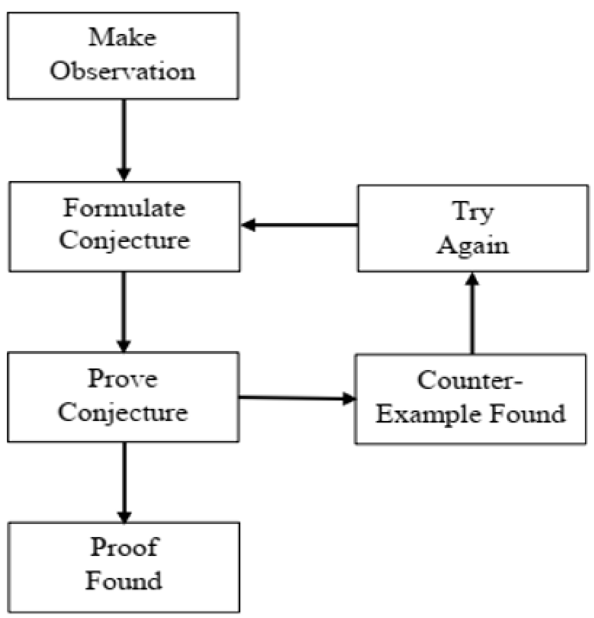


**Figure 1.**
Mathematical method.

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In the next section, we present some results of the previous studies as preliminaries. We then extend the results of previous studies on the generalizations of factoriangular numbers to provide some explorations on the further generalizations of factoriangular numbers as the main results.

## 3. Results and Discussion

### *3.1. Preliminaries*

A number that is a sum of a factorial and its corresponding triangular number is referred to as factoriangular number [10]. The factoriangular number is formally defined as follows:
Definition 3.1: The *n*th factoriangular number is defined by the formula

$$Ft_n = n! + T_n$$

where $n! = 1 \cdot 2 \cdot 3 \cdots n$ and $T_n = 1 + 2 + 3 + \ldots + n = n(n+1)/2$.

The first few factoriangular numbers are given in the sequence $\{2, 5, 12, 34, 135, 741, 5068, 40356, 362925, 3628855, \ldots\}$. This sequence, with 2 (i.e., $1! + T_1$) as the first term, appeared in OEIS as A101292 in 2004. In 2016, 1 (i.e., $0! + T_0$) was appended as the first term [13]. We call the numbers in this sequence *n*-factoriangular numbers.

The terms of the sequence of factoriangular numbers $\{Ft_n\}$ for natural numbers $n \geq 1$ are of the form
$Ft_n = (1 \cdot 2 \cdot 3 \cdots n) + (1 + 2 + 3 + \ldots + n)$.
There are several ways of generalizing this sequence.

A generalization of the sequence of *n*-factoriangular numbers [19] is the sequence $\{Ft_{n,k}\}$ for natural numbers $n, k \geq 1$, which follow the form

$$Ft_{n,k} = (1 \cdot 2 \cdot 3 \cdots n) + (1 + 2 + 3 + \ldots + k)$$

We call the numbers in this sequence $(n, k)$-factoriangular numbers and define as follows:
Definition 3.2: The $(n, k)$-factoriangular number is defined by the formula

$$Ft_{n,k} = n! + T_k$$

where $n! = 1 \cdot 2 \cdot 3 \cdots n$ and $T_k = 1 + 2 + 3 + \ldots + k = k(k+1)/2$ for natural numbers $n, k \geq 1$.

From Definitions 3.1 and 3.2, when $n = k$, the $(n, k)$-factoriangular numbers are the same as the *n*-factoriangular numbers or $Ft_{n,k} = Ft_n$.

The sequence of the $(n, k)$-factoriangular numbers is given by

$$\{Ft_{n,k}\} = \{2, 3, 4, 5, 7, 7, 9, 8, 12, 25, 11, 27, 12, 30, 16, 34, \ldots\}$$

for $(n,k)$ = (1,1), (2,1), (1,2), (2,2), (3,1), (1,3), (3,2), (2,3), (3,3), (4,1), (1,4), (4,2), (2,4), (4,3), (3,4), (4,4), …

Notice that in this sequence of $(n, k)$-factoriangular numbers, the term in the 1st, 4th, 9th, 16th, and so on are the *n*-factoriangular numbers. These terms are the entries in the main diagonal from the top-left to the bottom-right (i.e, when $n = k$) of the following table:

**Table 1.**
Table of (n,k)-factoriangular numbers.

| $n \setminus k$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | ⋯ | $k$ |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 2 | 4 | 7 | 11 | 16 | 22 | 29 | ⋯ | $1 + T_k$ |
| 2 | 3 | 5 | 8 | 12 | 17 | 23 | 30 | ⋯ | $2 + T_k$ |
| 3 | 7 | 9 | 12 | 16 | 21 | 27 | 34 | ⋯ | $6 + T_k$ |
| 4 | 25 | 27 | 30 | 34 | 39 | 45 | 52 | ⋯ | $24 + T_k$ |
| 5 | 121 | 123 | 126 | 130 | 135 | 141 | 148 | ⋯ | $120 + T_k$ |
| 6 | 721 | 723 | 726 | 730 | 735 | 741 | 748 | ⋯ | $720 + T_k$ |
| 7 | 5041 | 5043 | 5046 | 5050 | 5055 | 5061 | 5068 | ⋯ | $5040 + T_k$ |
| ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋱ | ⋮ |
| $n$ | $n! + 1$ | $n! + 3$ | $n! + 6$ | $n! + 10$ | $n! + 15$ | $n! + 21$ | $n! + 28$ | ⋯ | $n! + T_k$ |


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The *n*-factoriangular numbers were also generalized into multiple factoriangular numbers [18] to have

$$F_t(n,k) = (n!)^k + \sum n^k$$

wherein, $\sum n^k = T_n(k)$. This is similar to a generalization of the sequence of *n*-factoriangular numbers [20] into the sequence $\{Ft_{n^{(m)}}\}$ for natural numbers $n, m \geq 1$, which follow the form

$$Ft_{n^{(m)}} = (1^m \cdot 2^m \cdot 3^m \cdots n^m) + (1^m + 2^m + 3^m + \ldots + n^m)$$

We call the numbers in this sequence as $n^{(m)}$-factoriangular numbers and define as follows:
Definition 3.3: The $n^{(m)}$-factoriangular number is defined by the formula

$$Ft_{n^{(m)}} = (n!)^m + S_m(n)$$

where $(n!)^m = 1^m \cdot 2^m \cdot 3^m \cdots n^m$ and $S_m(n) = 1^m + 2^m + 3^m + \ldots + n^m$ for natural numbers $n, m \geq 1$.

From the Definitions 3.1 and 3.3, when $m = 1$, the $n^{(m)}$-factoriangular numbers are the same as the *n*-factoriangular numbers or $Ft_{n^{(m)}} = Ft_n$. Here, $S_1(n)$ is the same as $T_n$, and thus,

$$Ft_{n^{(1)}} = n! + S_1(n) = (1 \cdot 2 \cdot 3 \cdots n) + (1 + 2 + 3 + \ldots + n) = n! + \frac{n(n+1)}{2} = Ft_n$$

For the next few specific cases of $n^{(m)}$-factoriangular numbers, that is when $m = 2, 3, 4, 5, 6, 7, 8, 9$, we have

$Ft_{n^{(2)}} = (n!)^2 + S_2(n) = (1^2 \cdot 2^2 \cdot 3^2 \cdots n^2) + (1^2 + 2^2 + 3^2 + \ldots + n^2)$
$Ft_{n^{(3)}} = (n!)^3 + S_3(n) = (1^3 \cdot 2^3 \cdot 3^3 \cdots n^3) + (1^3 + 2^3 + 3^3 + \ldots + n^3)$
$Ft_{n^{(4)}} = (n!)^4 + S_4(n) = (1^4 \cdot 2^4 \cdot 3^4 \cdots n^4) + (1^4 + 2^4 + 3^4 + \ldots + n^4)$
$Ft_{n^{(5)}} = (n!)^5 + S_5(n) = (1^5 \cdot 2^5 \cdot 3^5 \cdots n^5) + (1^5 + 2^5 + 3^5 + \ldots + n^5)$
$Ft_{n^{(6)}} = (n!)^6 + S_6(n) = (1^6 \cdot 2^6 \cdot 3^6 \cdots n^6) + (1^6 + 2^6 + 3^6 + \ldots + n^6)$
$Ft_{n^{(7)}} = (n!)^7 + S_7(n) = (1^7 \cdot 2^7 \cdot 3^7 \cdots n^7) + (1^7 + 2^7 + 3^7 + \ldots + n^7)$
$Ft_{n^{(8)}} = (n!)^8 + S_8(n) = (1^8 \cdot 2^8 \cdot 3^8 \cdots n^8) + (1^5 + 2^8 + 3^8 + \ldots + n^8)$
$Ft_{n^{(9)}} = (n!)^9 + S_9(n) = (1^9 \cdot 2^9 \cdot 3^9 \cdots n^9) + (1^9 + 2^9 + 3^9 + \ldots + n^9)$

Theorem 3.4: For natural number $n \geq 1$, the $n^{(2)}$-, $n^{(3)}$-, $n^{(4)}$-, $n^{(5)}$-, $n^{(6)}$-, $n^{(7)}$-, $n^{(8)}$-, and $n^{(9)}$-factoriangular numbers are, respectively, given by the formulas

$Ft_{n^{(2)}} = (n!)^2 + \frac{1}{3}(2n+1)T_n$
$Ft_{n^{(3)}} = (n!)^3 + T_n^2$
$Ft_{n^{(4)}} = (n!)^4 + \frac{1}{15}(6n^3 + 9n^2 + n - 1)T_n$
$Ft_{n^{(5)}} = (n!)^5 + \frac{1}{3}(2n^2 + 2n - 1)T_n^2$
$Ft_{n^{(6)}} = (n!)^6 + \frac{1}{21}(6n^5 + 15n^4 + 6n^3 - 6n^2 - n + 1)T_n$
$Ft_{n^{(7)}} = (n!)^7 + \frac{1}{6}(3n^4 + 6n^3 - n^2 - 4n + 2)T_n^2$
$Ft_{n^{(8)}} = (n!)^8 + \frac{1}{45}(10n^7 + 35n^6 + 25n^5 - 25n^4 - 17n^3 + 17n^2 + 3n - 3)T_n$
$Ft_{n^{(9)}} = (n!)^9 + \frac{1}{5}(2n^6 + 6n^5 + n^4 - 8n^3 + n^2 + 6n - 3)T_n^2$

where $n! = 1 \cdot 2 \cdot 3 \cdots n$ and $T_n = 1 + 2 + 3 + \ldots + n = n(n+1)/2$.

A recent paper [20] gives the proof of Theorem 3.4. The proof has verified the following:

$(n+1)^2 - 1 = 2S_1 + n$
$(n+1)^3 - 1 = 3S_2 + 3S_1 + n$

$(n+1)^4 - 1 = 4S_3 + 6S_2 + 4S_1 + n$
$(n+1)^5 - 1 = 5S_4 + 10S_3 + 10S_2 + 5S_1 + n$
$(n+1)^6 - 1 = 6S_5 + 15S_4 + 20S_3 + 15S_2 + 6S_1 + n$
$(n+1)^7 - 1 = 7S_6 + 21S_5 + 35S_4 + 35S_3 + 21S_2 + 7S_1 + n$
$(n+1)^8 - 1 = 8S_7 + 28S_6 + 56S_5 + 70S_4 + 56S_3 + 28S_2 + 8S_1 + n$
$(n+1)^9 - 1 = 9S_8 + 36S_7 + 84S_6 + 126S_5 + 126S_4 + 84S_3 + 36S_2 + 9S_1 + n$
$(n+1)^{10} - 1 = 10S_9 + 45S_8 + 120S_7 + 210S_6 + 252S_5 + 210S_4 + 120S_3 + 45S_2 + 10S_1 + n$
For ease of writing of the above and to avoid confusion as to whether a function or a multiplication, the sum of powers of natural numbers *n* has been written simply as $S_m$ instead of $S_m(n)$ (e.g., $S_1$ instead of $S_1(n)$).

The sequences of $n^{(m)}$-factoriangular numbers, for some specific $m \geq 1$, are given as follows:
$\{Ft_{n^{(1)}}\} = \{2, 5, 12, 34, 135, \ldots\}$ for $m = 1$
$\{Ft_{n^{(2)}}\} = \{2, 9, 50, 606, 14455, \ldots\}$ for $m = 2$
$\{Ft_{n^{(3)}}\} = \{2, 17, 252, 13924, 1728225, \ldots\}$ for $m = 3$
$\{Ft_{n^{(4)}}\} = \{2, 33, 1394, 332130, 207360979, \ldots\}$ for $m = 4$
$\{Ft_{n^{(5)}}\} = \{2, 65, 8052, 7963924, 24883204425, \ldots\}$ for $m = 5$
$\{Ft_{n^{(6)}}\} = \{2, 129, 47450, 191107866, 2985984020515, \ldots\}$ for $m = 6$
$\{Ft_{n^{(7)}}\} = \{2, 257, 282252, 4586490124, 358318080096825, \ldots\}$ for $m = 7$
$\{Ft_{n^{(8)}}\} = \{2, 513, 1686434, 11062023330, 1334357900462979, \ldots\}$ for $m = 8$
$\{Ft_{n^{(9)}}\} = \{2, 1025, 10097892, 286607822564, 559780352002235465, \ldots\}$ for $m = 9$

We present the next theorem for the general case of the $n^{(m)}$-factoriangular numbers.
Theorem 3.5: For natural numbers $n, m \geq 1$, the $n^{(m)}$-factoriangular numbers can be determined by the formula

$$Ft_{n^{(m)}} = (n!)^m + \frac{1}{m+1}\left[(n+1)[(n+1)^m - 1] - \sum_{i=1}^{m-1} \binom{m+1}{i} S_i(n)\right]$$

where $n! = 1 \cdot 2 \cdot 3 \cdots n$ and $S_i(n) = 1^i + 2^i + 3^i + \ldots + n^i$.

A recent paper [18] gives the proof of Theorem 3.5. The proof has verified the following:

$$(n+1)^{m+1} - 1 = \binom{m+1}{m} S_m + \binom{m+1}{m-1} S_{m-1} + \binom{m+1}{m-2} S_{m-2} + \ldots + \binom{m+1}{1} S_1 + n$$

wherein, the $S_m(n)$ is simply written again as $S_m$.

We present the next two theorems for the even and odd *m* in the $n^{(m)}$-factoriangular numbers.
Theorem 3.6: The $n^{(m)}$-factoriangular number for even $m = 2k$ is given by the formula

$$Ft_{n^{(m)}} = Ft_{n^{(2k)}} = (n!)^{2k} + \frac{2n+1}{2k+1}[n^{2k-2} + P(n^{2k-3})]T_n$$

where $n! = 1 \cdot 2 \cdot 3 \cdots n$, $T_n = 1 + 2 + 3 + \ldots + n = n(n+1)/2$, and $P(n^{2k-3})$ is a polynomial in $n$ of degree $2k - 3$, for natural numbers $n, k \geq 1$.

A recent paper [20] gives the proof of Theorem 3.6. The proof has verified the following:

$$S_{2(1)} \quad = \quad S_2 = \frac{2n+1}{3} T_n$$
$$S_{2(2)} \quad = \quad S_4 = \frac{2n+1}{5}[n^2 + (n - \frac{1}{3})]T_n$$
$$S_{2(3)} \quad = \quad S_6 = \frac{2n+1}{7}[n^4 + (2n^3 - n + \frac{1}{3})]T_n$$
$$S_{2(4)} \quad = \quad S_8 = \frac{2n+1}{9}[n^6 + (3n^5 + n^4 - 3n^3 - \frac{1}{5}n^2 + \frac{9}{5}n - \frac{3}{5})]T_n$$


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$S_{2k} = \dfrac{2n+1}{2k+1}[n^{2k-2} + P(n^{2k-3})]T_n$

wherein, the $S_m(n)$ is simply written again as $S_m$ (e.g., $S_2$ means $S_2(n)$).

Theorem 3.7: The $n^{(m)}$-factoriangular number for odd $m = 2k+1$ is given by the formula

$$Ft_{n^{(m)}} = Ft_{n^{(2k+1)}} = (n!)^{2k+1} + \frac{n(n+1)}{k+1}[n^{2k-2} + P(n^{2k-3})]T_n$$

where $n! = 1 \cdot 2 \cdot 3 \cdots n$, $T_n = 1 + 2 + 3 + \ldots + n = n(n+1)/2$, and $P(n^{2k-3})$ is a polynomial in $n$ of degree $2k-3$, for natural numbers $n, k \geq 1$.

A recent paper [20] gives the proof of Theorem 3.7. The proof has verified the following:

$$S_{2(1)+1} = S_3 = \frac{n(n+1)}{2}T_n$$
$$S_{2(2)+1} = S_5 = \frac{n(n+1)}{3}[n^2 + (n - \frac{1}{2})]T_n$$
$$S_{2(3)+1} = S_7 = \frac{n(n+1)}{4}[n^4 + (2n^3 - \frac{1}{3}n^2 - \frac{4}{3}n + \frac{2}{3})]T_n$$
$$S_{2(4)+1} = S_9 = \frac{n(n+1)}{5}[n^6 + (3n^5 + \frac{1}{2}n^4 - 4n^3 + \frac{1}{2}n^2 + 3n - \frac{3}{2})]T_n$$
$$S_{2k+1} = \frac{n(n+1)}{k+1}[n^{2k-2} + P(n^{2k-3})]T_n$$

wherein, the $S_m(n)$ is simply written again as $S_m$ (e.g., $S_3$ means $S_3(n)$).

We also present the next two theorems for the even and odd $m$ in the $n^{(m)}$-factoriangular numbers that involves representation in the sum of powers instead of the triangular numbers.

Theorem 3.8: The $n^{(m)}$-factoriangular number for even $m = 2k$ is given by the formula

$$Ft_{n^{(m)}} = Ft_{n^{(2k)}} = (n!)^{2k} + \frac{3}{2k+1}[n^{2k-2} + P(n^{2k-3})]S_2(n)$$

where $n! = 1 \cdot 2 \cdot 3 \cdots n$, $S_2(n) = n(n+1)(2n+1)/6$, $S_3(n) = n^2(n+1)^2/4$, and $P(n^{2k-3})$ is a polynomial in $n$ of degree $2k-3$, for natural numbers $n \geq 1$and $k > 1$.

A recent paper [20] gives the proof of Theorem 3.8. The proof has verified the following:

$$S_{2(2)} = S_4 = \frac{3}{5}[n^2 + (n - \frac{1}{3})]S_2$$
$$S_{2(3)} = S_6 = \frac{3}{7}[n^4 + (2n^3 - n + \frac{1}{3})]S_2$$
$$S_{2(4)} = S_8 = \frac{3}{9}[n^6 + (3n^5 + n^4 - 3n^3 - \frac{1}{5}n^2 + \frac{9}{5}n - \frac{3}{5})]S_2$$
$$S_{2k} = \frac{3}{2k+1}[n^{2k-2} + P(n^{2k-3})]S_2$$

Where in, the $S_m(n)$ is simply written again as $S_m$ (e.g., $S_4$ means $S_4(n)$).

Theorem 3.9: The $n^{(m)}$-factoriangular number for odd $m = 2k+1$ is given by the formula

$$Ft_{n^{(m)}} = Ft_{n^{(2k+1)}} = (n!)^{2k+1} + \frac{2}{k+1}[n^{2k-2} + P(n^{2k-3})]S_3(n)$$

where $n! = 1 \cdot 2 \cdot 3 \cdots n$, $S_2(n) = n(n+1)(2n+1)/6$, $S_3(n) = n^2(n+1)^2/4$, and $P(n^{2k-3})$ is a polynomial in $n$ of degree $2k-3$, for natural numbers $n \geq 1$and $k > 1$.

A recent paper [20] gives the proof of Theorem 3.9. The proof has verified the following:

$$S_{2(2)+1} = S_5 = \frac{2}{3}[n^2 + (n - \frac{1}{2})]S_3$$
$$S_{2(3)+1} = S_7 = \frac{2}{4}[n^4 + (2n^3 - \frac{1}{3}n^2 - \frac{4}{3}n + \frac{2}{3})]S_3$$


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$S_{2(4)+1} \;=\; S_8 = \frac{2}{5}[n^6 + (3n^5 + \frac{1}{2}n^4 - 4n^3 + \frac{1}{2}n^2 + 3n - \frac{3}{2})]S_3$

$S_{2k+1} = \frac{2}{k+1}[n^{2k-2} + P(n^{2k-3})]S_3$

Where in, the $S_m(n)$ is simply written again as $S_m$ (e.g., $S_5$ means $S_5(n)$).

The $\{Ft_{n,k}\}$ and $\{Ft_{n^{(m)}}\}$ can be combined to produce another generalization of the sequence of *n*-factoriangular numbers, which is the sequence } for natural numbers $n, k, m \geq 1$ that follow the form

$$Ft_{n^{(m)},k^{(m)}} = (1^m \cdot 2^m \cdot 3^m \cdots n^m) + (1^m + 2^m + 3^m + \ldots + k^m)$$

We call the numbers in this sequence as $(n^{(m)}, k^{(m)})$-factoriangular numbers and define as follows:

Definition 3.10: The $(n^{(m)}, k^{(m)})$-factoriangular number is defined by the formula

$$Ft_{n^{(m)},k^{(m)}} = (n!)^m + S_m(k)$$

where $(n!)^m = 1^m \cdot 2^m \cdot 3^m \cdots n^m$ and $S_m(k) = 1^m + 2^m + 3^m + \ldots + k^m$ for natural numbers $n, k, m \geq 1$.

From the above definitions, when $m = 1$, the $(n^{(m)}, k^{(m)})$-factoriangular numbers are the same as the $(n, k)$-factoriangular numbers or $Ft_{n^{(m)},k^{(m)}} = Ft_{n,k}$; when $n = k$, the $(n^{(m)}, k^{(m)})$-factoriangular numbers are the same as the $n^{(m)}$-factoriangular numbers or $Ft_{n^{(m)},k^{(m)}} = Ft_{n^{(m)}}$; and $m = 1$ and $n = k$, the $(n^{(m)}, k^{(m)})$-factoriangular numbers are the same as the *n*-factoriangular numbers or $Ft_{n^{(m)},k^{(m)}} = Ft_n$.

The sequence of $(n^{(m)}, k^{(m)})$-factoriangular numbers can be further generalized into the sequence $\{Ft_{n^{(a)},k^{(b)}}\}$ for natural numbers $n, k, a, b \geq 1$, which follow the form

$$Ft_{n^{(a)},k^{(b)}} = (1^a \cdot 2^a \cdot 3^a \cdots n^a) + (1^b + 2^b + 3^b + \ldots + k^b)$$

We call the numbers in this sequence as $(n^{(a)}, k^{(b)})$-factoriangular numbers and define as follows:

Definition 3.11: The $(n^{(a)}, k^{(b)})$-factoriangular number is defined by the formula

$$Ft_{n^{(a)},k^{(b)}} = (n!)^a + S_b(k)$$

where $(n!)^a = 1^a \cdot 2^a \cdot 3^a \cdots n^a$ and $S_b(k) = 1^b + 2^b + 3^b + \ldots + k^b$ for natural numbers $n, k, a, b \geq 1$.

From the above definitions, when $a = b = m$, the $(n^{(a)}, k^{(b)})$-factoriangular numbers are the same as the $(n^{(m)}, k^{(m)})$-factoriangular numbers or $Ft_{n^{(a)},k^{(b)}} = Ft_{n^{(m)},k^{(m)}}$; when $a = b = 1$, the $(n^{(a)}, k^{(b)})$-factoriangular numbers are the same as the $(n, k)$-factoriangular numbers or $Ft_{n^{(a)},k^{(b)}} = Ft_{n,k}$; when $n = k$ and $a = b = m$, the $(n^{(a)}, k^{(b)})$-factoriangular numbers are the same as the $n^{(m)}$-factoriangular numbers or $Ft_{n^{(a)},k^{(b)}} = Ft_{n^{(m)}}$; and when $n = k$ and $a = b = 1$, the $(n^{(a)}, k^{(b)})$-factoriangular numbers are the same as the *n*-factoriangular numbers or $Ft_{n^{(a)},k^{(b)}} = Ft_n$.

The above definitions were taken from previous study Castillo [20]. However, the said study focuses only on the $n^{(m)}$-factoriangular numbers. In the present study, we focus on the $(n^{(m)}, k^{(m)})$-factoriangular numbers and on the $(n^{(a)}, k^{(b)})$-factoriangular numbers.

### *3.2. Main Results*

#### *3.2.1. On the $(n^{(m)}, k^{(m)})$-Factoriangular Numbers*

We integrate the notions of $(n, k)$-factoriangular numbers and $n^{(m)}$-factoriangular numbers to form the $(n^{(m)}, k^{(m)})$-factoriangular numbers (see Definition 3.10). We now prove the succeeding theorems and give examples of sequences of $(n^{(m)}, k^{(m)})$-factoriangular numbers. For simplicity of notation and to avoid confusion between a function and a multiplication, we use $S_m$ in lieu of $S_m(k)$ in the proofs.

Theorem 3.12: For natural numbers $n, k, m \geq 1$, the $(n^{(m)}, k^{(m)})$-factoriangular numbers can be determined by the formula

$$Ft_{n^{(m)},k^{(m)}} = (n!)^m + \frac{1}{m+1}\left[(k+1)[(k+1)^m - 1] - \sum_{i=1}^{m-1}\binom{m+1}{i}S_i(k)\right]$$

where $n! = 1 \cdot 2 \cdot 3 \cdots n$ and $S_i(k) = 1^i + 2^i + 3^i + \ldots + k^i$.

Proof: From Definition 3.10, $Ft_{n^{(m)},k^{(m)}} = (n!)^m + S_m(k)$. What we need to show is that

$$S_m(k) = \frac{1}{m+1}\left[(k+1)[(k+1)^m - 1] - \sum_{i=1}^{m-1}\binom{m+1}{i}S_i(k)\right]$$

From a previous paper [20] we deduce that

$$(k+1)^{m+1} - 1 = \binom{m+1}{m}S_m + \binom{m+1}{m-1}S_{m-1} + \binom{m+1}{m-2}S_{m-2} + \ldots + \binom{m+1}{1}S_1 + k$$

or

$$(k+1)^{m+1} - (k+1) = (m+1)S_m + \binom{m+1}{1}S_1 + \binom{m+1}{2}S_2 + \ldots + \binom{m+1}{m-1}S_{m-1}$$

and then,

$$S_m = \frac{1}{m+1}\left[(k+1)[(k+1)^m - 1] - \sum_{i=1}^{m-1}\binom{m+1}{i}S_i\right]$$

It follows shortly that

$$Ft_{n^{(m)},k^{(m)}} = (n!)^m + \frac{1}{m+1}\left[(k+1)[(k+1)^m - 1] - \sum_{i=1}^{m-1}\binom{m+1}{i}S_i(k)\right]$$

Theorem 3.13: The $(n^{(m)}, k^{(m)})$-factoriangular number for even $m = 2j$ is given by the formula

$$Ft_{n^{(m)},k^{(m)}} = Ft_{n^{(2j)},k^{(2j)}} = (n!)^{2j} + \frac{2k+1}{2j+1}\left[k^{2j-2} + P(k^{2j-3})\right]T_k$$

where $n! = 1 \cdot 2 \cdot 3 \cdots n$, $T_k = 1 + 2 + 3 + \ldots + k = k(k+1)/2$, and $P(k^{2j-3})$ is a polynomial in $k$ of degree $2j-3$, for natural numbers $n, k, j \geq 1$.

Proof: From Definition 3.10, $Ft_{n^{(m)},k^{(m)}} = (n!)^m + S_m(k)$. We need to show that

$$S_m(k) = S_{2j}(k) = \frac{2k+1}{2j+1}\left[k^{2j-2} + P(k^{2j-3})\right]T_k$$

It is similar to a previous proof [20] that

$$S_{2k}(n) = \frac{2n+1}{2k+1}[n^{2k-2} + P(n^{2k-3})]T_n$$

we simply have

$$S_{2j}(k) = \frac{2k+1}{2j+1}\left[k^{2j-2} + P(k^{2j-3})\right]T_k$$

and then

$$Ft_{n^{(m)},k^{(m)}} = Ft_{n^{(2j)},k^{(2j)}} = (n!)^{2j} + \frac{2k+1}{2j+1}\left[k^{2j-2} + P(k^{2j-3})\right]T_k$$

Theorem 3.14: The $(n^{(m)}, k^{(m)})$-factoriangular number for odd $m = 2j+1$ is given by the formula

$$Ft_{n^{(m)},k^{(m)}} = Ft_{n^{(2j+1)},k^{(2j+1)}} = (n!)^{2j+1} + \frac{k(k+1)}{j+1}\left[k^{2j-2} + P(k^{2j-3})\right]T_k$$

where $n! = 1 \cdot 2 \cdot 3 \cdots n$, $T_k = 1 + 2 + 3 + \ldots + k = k(k+1)/2$, and $P(k^{2j-3})$ is a polynomial in $k$ of degree $2j-3$, for natural numbers $n, k, j \geq 1$.

Proof: From Definition 3.10, $Ft_{n^{(m)},k^{(m)}} = (n!)^m + S_m(k)$. We need to show that

$$S_m(k) = S_{2j+1}(k) = \frac{k(k+1)}{j+1}\left[k^{2j-2} + P(k^{2j-3})\right]T_k$$

It is similar to a previous proof [20] that

$$S_{2k+1}(n) = \frac{n(n+1)}{k+1}[n^{2k-2} + P(n^{2k-3})]T_n$$

we also have

$$S_{2j+1}(k) = \frac{k(k+1)}{j+1}\left[k^{2j-2} + P(k^{2j-3})\right]T_k$$

Hence,

$$Ft_{n^{(m)},k^{(m)}} = Ft_{n^{(2j+1)},k^{(2j+1)}} = (n!)^{2j+1} + \frac{k(k+1)}{j+1}\left[k^{2j-2} + P(k^{2j-3})\right]T_k$$

Theorem 3.15: The $(n^{(m)}, k^{(m)})$-factoriangular number for even $m = 2j$ is given by the formula

$$Ft_{n^{(m)},k^{(m)}} = Ft_{n^{(2j)},k^{(2j)}} = (n!)^{2j} + \frac{3}{2j+1}\left[k^{2j-2} + P(k^{2j-3})\right]S_2(k)$$

where $n! = 1 \cdot 2 \cdot 3 \cdots n$, $S_2(k) = k(k+1)(2k+1)/6$, and $P(k^{2j-3})$ is a polynomial in $k$ of degree $2j-3$, for natural numbers $n, k \geq 1$and $j > 1$.

Proof: From Definition 3.10, $Ft_{n^{(m)},k^{(m)}} = (n!)^m + S_m(k)$. We need to show that

$$S_m(k) = S_{2j}(k) = \frac{3}{2j+1}\left[k^{2j-2} + P(k^{2j-3})\right]S_2(k)$$

It is similar to a previous proof [20] that

$$S_{2k}(n) = \frac{3}{2k+1}\left[n^{2k-2} + P\left(n^{2k-3}\right)\right]S_2(n)$$

we have

$$S_{2j}(k) = \frac{3}{2j+1}\left[k^{2j-2} + P(k^{2j-3})\right]S_2(k)$$

and then

$$Ft_{n^{(m)},k^{(m)}} = Ft_{n^{(2j)},k^{(2j)}} = (n!)^{2j} + \frac{3}{2j+1}\left[k^{2j-2} + P(k^{2j-3})\right]S_2(k)$$

Theorem 3.16: The $(n^{(m)}, k^{(m)})$-factoriangular number for odd $m = 2j+1$ is given by the formula

$$Ft_{n^{(m)},k^{(m)}} = Ft_{n^{(2j+1)},k^{(2j+1)}} = (n!)^{2j+1} + \frac{2}{j+1}\left[k^{2j-2} + P(k^{2j-3})\right]S_3(k)$$

where $n! = 1 \cdot 2 \cdot 3 \cdots n$, $S_3(k) = k^2(k+1)^2/4$, and $P(k^{2j-3})$ is a polynomial in $k$ of degree $2j-3$, for natural numbers $n, k \geq 1$and $j > 1$.

Proof: From Definition 3.10, $Ft_{n^{(m)},k^{(m)}} = (n!)^m + S_m(k)$. We need to show that

$$S_m(k) = S_{2j+1}(k) = \frac{2}{j+1}\left[k^{2j-2} + P(k^{2j-3})\right]S_3(k)$$

It is similar to a previous proof [20] that

$$S_{2k+1}(m) = \frac{2}{k+1}\left[n^{2k-2} + P\left(n^{2k-3}\right)\right]S_3(m)$$

we have

$$S_{2j+1}(k) = \frac{2}{j+1}\left[k^{2j-2} + P(k^{2j-3})\right]S_3(k)$$

Thus,


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$$Ft_{n^{(m)},k^{(m)}} = Ft_{n^{(2j+1)},k^{(2j+1)}} = (n!)^{2j+1} + \frac{2}{j+1}\left[k^{2j-2} + P(k^{2j-3})\right]S_3(k)$$

Two examples of $Ft_{n^{(m)},k^{(m)}}$, with $m > 1$, are presented here. The first example is when $m = 2$ and the $(n^{(2)}, k^{(2)})$-factoriangular numbers are given in Table 2.

**Table 2.**
Table of $(n^{(2)}, k^{(2)})$-factoriangular numbers.

| $n \setminus k$ | 1 | 2 | 3 | 4 | 5 | ⋯ | $k$ |
|---|---|---|---|---|---|---|---|
| 1 | 2 | 6 | 15 | 31 | 56 | ⋯ | $1 + S_2(k)$ |
| 2 | 5 | 9 | 18 | 34 | 59 | ⋯ | $4 + S_2(k)$ |
| 3 | 37 | 41 | 50 | 66 | 91 | ⋯ | $36 + S_2(k)$ |
| 4 | 577 | 581 | 590 | 606 | 631 | ⋯ | $576 + S_2(k)$ |
| 5 | 14401 | 14405 | 14414 | 14430 | 14455 | ⋯ | $14400 + S_2(k)$ |
| ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋱ | ⋮ |
| $n$ | $(n!)^2 + 1$ | $(n!)^2 + 5$ | $(n!)^2 + 14$ | $(n!)^2 + 30$ | $(n!)^2 + 55$ | ⋯ | $(n!)^2 + S_2(k)$ |

Then, the sequence of $(n^{(2)}, k^{(2)})$-factoriangular numbers is given by

$$\{Ft_{n^{(2)},k^{(2)}}\} = \{2, 5, 6, 9, 37, 15, 41, 18, 50, \ldots\}$$

for $(n,k) = (1,1), (2,1), (1,2), (2,2), (3,1), (1,3), (3,2), (2,3), (3,3), \ldots$ .

The second example is when $m = 3$ and the $(n^{(3)}, k^{(3)})$-factoriangular numbers are given in Table 3.

**Table 3.**
Table of $(n^{(3)}, k^{(3)})$-factoriangular numbers.

| $n \setminus k$ | 1 | 2 | 3 | 4 | ⋯ | $k$ |
|---|---|---|---|---|---|---|
| **1** | 2 | 10 | 37 | 101 | ⋯ | $1 + S_3(k)$ |
| **2** | 9 | 17 | 44 | 108 | ⋯ | $8 + S_3(k)$ |
| **3** | 217 | 225 | 252 | 316 | ⋯ | $216 + S_3(k)$ |
| **4** | 13825 | 13833 | 13860 | 13924 | ⋯ | $13824 + S_3(k)$ |
| ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋱ | ⋮ |
| $n$ | $(n!)^3 + 1$ | $(n!)^3 + 9$ | $(n!)^3 + 36$ | $(n!)^3 + 100$ | ⋯ | $(n!)^3 + S_3(k)$ |

Then, the sequence of $(n^{(3)}, k^{(3)})$-factoriangular numbers is given by

$$\{Ft_{n^{(3)},k^{(3)}}\} = \{2, 9, 10, 17, 217, 37, 225, 44, 252, \ldots\}$$

for $(n,k) = (1,1), (2,1), (1,2), (2,2), (3,1), (1,3), (3,2), (2,3), (3,3), \ldots$ .

*3.2.2. On the $(n^{(a)}, k^{(b)})$-Factoriangular Numbers*

We further generalized $(n^{(m)}, k^{(m)})$-factoriangular numbers to have the $(n^{(a)}, k^{(b)})$-factoriangular numbers (see Definition 3.11). We now present the following theorems whose proofs are similar to the previous theorems on $(n^{(m)}, k^{(m)})$-factoriangular numbers. We also give examples of sequences of $(n^{(a)}, k^{(b)})$-factoriangular numbers.

Theorem 3.17: For natural numbers $n, k, a, b \geq 1$, the $(n^{(a)}, k^{(b)})$-factoriangular numbers can be determined by the formula

$$Ft_{n^{(a)},k^{(b)}} = (n!)^a + \frac{1}{b+1}\left[(k+1)[(k+1)^b - 1] - \sum_{i=1}^{b-1}\binom{b+1}{i}S_i(k)\right]$$

where $n! = 1 \cdot 2 \cdot 3 \cdots n$ and $S_i(k) = 1^i + 2^i + 3^i + \ldots + k^i$.

The proof of Theorem 3.17 is similar to the proof of Theorem 3.12.

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Theorem 3.18: The $(n^{(a)}, k^{(b)})$-factoriangular number for even $b = 2j$ is given by the formula

$$Ft_{n^{(a)},k^{(b)}} = Ft_{n^{(a)},k^{(2j)}} = (n!)^a + \frac{2k+1}{2j+1}\left[k^{2j-2} + P(k^{2j-3})\right]T_k$$

where $n! = 1 \cdot 2 \cdot 3 \cdots n$, $T_k = 1 + 2 + 3 + \ldots + k = k(k+1)/2$, and $P(k^{2j-3})$ is a polynomial in $k$ of degree $2j - 3$, for natural numbers $n, k, j \geq 1$.

The proof of Theorem 3.18 is similar to the proof of Theorem 3.13.

Theorem 3.19: The $(n^{(a)}, k^{(b)})$-factoriangular number for odd $b = 2j + 1$ is given by the formula

$$Ft_{n^{(a)},k^{(b)}} = Ft_{n^{(a)},k^{(2j+1)}} = (n!)^a + \frac{k(k+1)}{j+1}\left[k^{2j-2} + P(k^{2j-3})\right]T_k$$

where $n! = 1 \cdot 2 \cdot 3 \cdots n$, $T_k = 1 + 2 + 3 + \ldots + k = k(k+1)/2$, and $P(k^{2j-3})$ is a polynomial in $k$ of degree $2j - 3$, for natural numbers $n, k, j \geq 1$.

The proof of Theorem 3.19 is similar to the proof of Theorem 3.14.

Theorem 3.20: The $(n^{(a)}, k^{(b)})$-factoriangular number for even $b = 2j$ is given by the formula

$$Ft_{n^{(a)},k^{(b)}} = Ft_{n^{(a)},k^{(2j)}} = (n!)^a + \frac{3}{2j+1}\left[k^{2j-2} + P(k^{2j-3})\right]S_2(k)$$

where $n! = 1 \cdot 2 \cdot 3 \cdots n$, $S_2(k) = k(k+1)(2k+1)/6$, and $P(k^{2j-3})$ is a polynomial in $k$ of degree $2j - 3$, for natural numbers $n, k \geq 1$and $j > 1$.

The proof of Theorem 3.20 is similar to the proof of Theorem 3.15.

Theorem 3.21: The $(n^{(a)}, k^{(b)})$-factoriangular number for odd $b = 2j + 1$ is given by the formula

$$Ft_{n^{(a)},k^{(b)}} = Ft_{n^{(a)},k^{(2j+1)}} = (n!)^a + \frac{2}{j+1}\left[k^{2j-2} + P(k^{2j-3})\right]S_3(k)$$

where $n! = 1 \cdot 2 \cdot 3 \cdots n$, $S_3(k) = k^2(k+1)^2/4$, and $P(k^{2j-3})$ is a polynomial in $k$ of degree $2j - 3$, for natural numbers $n, k \geq 1$and $j > 1$.

The proof of Theorem 3.21 is similar to the proof of Theorem 3.16.

Three examples of $Ft_{n^{(a)},k^{(b)}}$, with $a, b \geq 1$ but not both $a, b = 1$, are presented here. When $a = 1$ and $b = 2$, the $(n^{(1)}, k^{(2)})$-factoriangular numbers are given in Table 4.

**Table 4.**
Table of $(n^{(1)}, k^{(2)})$-factoriangular numbers.

| $n \setminus k$ | 1 | 2 | 3 | 4 | 5 | $\cdots$ | $k$ |
|---|---|---|---|---|---|---|---|
| **1** | 2 | 6 | 15 | 31 | 56 | $\cdots$ | $1 + S_2(k)$ |
| **2** | 3 | 7 | 16 | 32 | 57 | $\cdots$ | $2 + S_2(k)$ |
| **3** | 7 | 11 | 20 | 36 | 61 | $\cdots$ | $6 + S_2(k)$ |
| **4** | 25 | 29 | 38 | 54 | 79 | $\cdots$ | $24 + S_2(k)$ |
| **5** | 121 | 125 | 134 | 150 | 175 | $\cdots$ | $120 + S_2(k)$ |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\ddots$ | $\vdots$ |
| $n$ | $n! + 1$ | $n! + 5$ | $n! + 14$ | $n! + 30$ | $n! + 55$ | $\cdots$ | $n! + S_2(k)$ |

Then, the sequence of $(n^{(1)}, k^{(2)})$-factoriangular numbers is given by

$$\left\{Ft_{n^{(1)},k^{(2)}}\right\} = \{2, 3, 6, 7, 7, 15, 11, 16, 20, \ldots\}$$

for $(n,k)$ = (1,1), (2,1), (1,2), (2,2), (3,1), (1,3), (3,2), (2,3), (3,3), … .

When $a = 2$ and $b = 1$, the $(n^{(2)}, k^{(1)})$-factoriangular numbers are given in Table 5.

**Table 5.**
Table of $(n^{(2)}, k^{(1)})$-factoriangular numbers.

| $n\backslash k$ | 1 | 2 | 3 | 4 | 5 | ⋯ | $k$ |
|---|---|---|---|---|---|---|---|
| 1 | 2 | 4 | 7 | 11 | 16 | ⋯ | $1 + S_1(k)$ |
| 2 | 5 | 7 | 10 | 14 | 19 | ⋯ | $4 + S_1(k)$ |
| 3 | 37 | 39 | 42 | 46 | 51 | ⋯ | $36 + S_1(k)$ |
| 4 | 577 | 579 | 582 | 586 | 591 | ⋯ | $576 + S_1(k)$ |
| 5 | 14401 | 14403 | 14406 | 14410 | 14415 | ⋯ | $14400 + S_1(k)$ |
| ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋱ | ⋮ |
| $n$ | $(n!)^2 + 1$ | $(n!)^2 + 3$ | $(n!)^2 + 6$ | $(n!)^2 + 10$ | $(n!)^2 + 15$ | ⋯ | $(n!)^2 + S_1(k)$ |

Then, the sequence of $(n^{(2)}, k^{(1)})$-factoriangular numbers is given by

$$\{Ft_{n^{(2)},k^{(1)}}\} = \{2, 5, 4, 7, 37, 7, 39, 10, 42, \dots\}$$

for $(n,k) = (1,1), (2,1), (1,2), (2,2), (3,1), (1,3), (3,2), (2,3), (3,3), \dots$ .

When $a = 2$ and $b = 3$, the $(n^{(2)}, k^{(3)})$-factoriangular numbers are given in Table 6.

**Table 6.**
Table of $(n^{(2)}, k^{(3)})$-factoriangular numbers

| $n\backslash k$ | 1 | 2 | 3 | 4 | 5 | ⋯ | $k$ |
|---|---|---|---|---|---|---|---|
| 1 | 2 | 10 | 37 | 101 | 226 | ⋯ | $1 + S_3(k)$ |
| 2 | 5 | 13 | 40 | 104 | 229 | ⋯ | $4 + S_3(k)$ |
| 3 | 37 | 45 | 72 | 136 | 261 | ⋯ | $36 + S_3(k)$ |
| 4 | 577 | 585 | 612 | 676 | 801 | ⋯ | $576 + S_3(k)$ |
| 5 | 14401 | 14409 | 14436 | 14500 | 14625 | ⋯ | $14400 + S_3(k)$ |
| ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋱ | ⋮ |
| $n$ | $(n!)^2 + 1$ | $(n!)^2 + 9$ | $(n!)^2 + 36$ | $(n!)^2 + 100$ | $(n!)^2 + 225$ | ⋯ | $(n!)^2 + S_3(k)$ |

Then, the sequence of $(n^{(2)}, k^{(3)})$-factoriangular numbers is given by

$$\{Ft_{n^{(2)},k^{(3)}}\} = \{2, 5, 10, 13, 37, 37, 45, 40, 72, \dots\}$$

for $(n,k) = (1,1), (2,1), (1,2), (2,2), (3,1), (1,3), (3,2), (2,3), (3,3), \dots$ .

## 4. Conclusions

Research on factoriangular numbers is still relatively new with its introduction to the number theory literature only in 2015. A factoriangular number is formed by adding a factorial and its additive analog, a triangular number. When a triangular number is added to its corresponding factorial, the result is an *n*-factoriangular number. The sequence $\{Ft_n\} = \{2, 5, 12, 34, 135, 741, 5068, \dots\}$ is the sequence of *n*-factoriangular numbers. The terms in the sequence can be generated by using the formula $Ft_n = n! + T_n$, where $n! = 1 \cdot 2 \cdot 3 \cdots n$ and $T_n = 1 + 2 + 3 + \dots + n$. This sequence of *n*-factoriangular numbers can be generalized in several ways. One generalization is the sequence of $(n, k)$-factoriangular numbers of the form $Ft_{n,k} = n! + T_k$, where $n! = 1 \cdot 2 \cdot 3 \cdots n$ and $T_k = 1 + 2 + 3 + \dots + k$. Another generalization is the sequence of $n^{(m)}$-factoriangular numbers of the form $Ft_{n^{(m)}} = (n!)^m + S_m(n)$, where $(n!)^m = 1^m \cdot 2^m \cdot 3^m \cdots n^m$ and $S_m(n) = 1^m + 2^m + 3^m + \dots + n^m$.

More generalizations can be made by integrating the concepts of $(n, k)$-factoriangular numbers and $n^{(m)}$-factoriangular numbers to produce the sequence of $(n^{(m)}, k^{(m)})$-factoriangular numbers of the form $Ft_{n^{(m)},k^{(m)}} = (n!)^m + S_m(k)$, where $(n!)^m = 1^m \cdot 2^m \cdot 3^m \cdots n^m$ and $S_m(k) = 1^m + 2^m + 3^m + \dots + k^m$. The sequence of $(n^{(m)}, k^{(m)})$-factoriangular numbers can be further generalized into the sequence of $(n^{(a)}, k^{(b)})$-factoriangular numbers of the form $Ft_{n^{(a)},k^{(b)}} = (n!)^a + S_b(k)$, where $(n!)^a = 1^a \cdot 2^a \cdot 3^a \cdots n^a$ and $S_b(k) = 1^b + 2^b + 3^b + \dots + k^b$.

The $(n^{(m)}, k^{(m)})$-factoriangular numbers can also be determined by the formula $Ft_{n^{(m)},k^{(m)}} = (n!)^m + \frac{1}{m+1}\left[(k+1)[(k+1)^m - 1] - \sum_{i=1}^{m-1}\binom{m+1}{i}S_i(k)\right]$ and the $(n^{(a)}, k^{(b)})$-factoriangular numbers by the formula $Ft_{n^{(a)},k^{(b)}} = (n!)^a + \frac{1}{b+1}\left[(k+1)[(k+1)^b - 1] - \sum_{i=1}^{b-1}\binom{b+1}{i}S_i(k)\right]$. The formulas for $(n^{(m)}, k^{(m)})$-factoriangular number for even $m = 2j$ are $Ft_{n^{(m)},k^{(m)}} = Ft_{n^{(2j)},k^{(2j)}} = (n!)^{2j} + \frac{2k+1}{2j+1}\left[k^{2j-2} + P(k^{2j-3})\right]T_k$ and $Ft_{n^{(m)},k^{(m)}} = Ft_{n^{(2j)},k^{(2j)}} = (n!)^{2j} + \frac{3}{2j+1}\left[k^{2j-2} + P(k^{2j-3})\right]S_2(k)$. The formulas for $(n^{(m)}, k^{(m)})$-factoriangular number for odd $m = 2j+1$ are $Ft_{n^{(m)},k^{(m)}} = Ft_{n^{(2j+1)},k^{(2j+1)}} = (n!)^{2j+1} + \frac{k(k+1)}{j+1}\left[k^{2j-2} + P(k^{2j-3})\right]T_k$ and $Ft_{n^{(m)},k^{(m)}} = Ft_{n^{(2j+1)},k^{(2j+1)}} = (n!)^{2j+1} + \frac{2}{j+1}\left[k^{2j-2} + P(k^{2j-3})\right]S_3(k)$. Similar formulas can be provided for $(n^{(a)}, k^{(b)})$-factoriangular number for even $m = 2j$ and for odd $m = 2j+1$.

Triangular arrays of factoriangular numbers may be formed from the tables of generalized factoriangular numbers. This will be of future interest to other mathematical explorers, especially those in the field of recreational mathematics.

## Transparency:

The author confirms that the manuscript is an honest, accurate, and transparent account of the study; that no vital features of the study have been omitted; and that any discrepancies from the study as planned have been explained. This study followed all ethical practices during writing.

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## References

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