Edelweiss Applied Science and Technology
ISSN: 2576-8484
Vol. 9, No. 5, 1393-1405
2025
Publisher: Learning Gate
DOI: 10.55214/25768484.v9i5.7172

Solution of neutrosophic fractional differential equations by neutrosophic extension principal method

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Abstract: This paper aims at a new approach for finding the solution of neutrosophic fuzzy fractional differential equations (NFFDEs) based on the Zadeh's Extension Principle method. NFFDEs combine fractional-order systems with uncertainty, which deals with truth, indeterminacy, and falsity information. This approach competently addresses the challenges modeled by both the fractional derivatives and the indeterminate constructions characteristic of neutrosophic systems. The paper frames the theoretical framework, advances the solution process, and validates the usefulness of the method. Theoretical and numerical results validate that the Extension Principle method conserves vital properties of the fundamental systems while providing flexible and inclusive representations of uncertainty.
Keywords: *Fractional derivative, Fractional differential equation, Fuzzy logic, Neutrosphic set theory.*

1. Introduction

Differential equation is very important techniques for theoretical Cooke [1] and Ross [2] as well as modelling Braun and Golubitsky [3] and Sobczyk [4] based study. If formed in continuous system. Several modifications and variation are already done in the wide field of research involving differential equations. In that context the order of any differential equation need not always be integer, it may be fractional order [5-8]. A fractional differential equation includes derivatives of non-integer (fractional) order which encompassing the concept of classical calculus. It models systems with memory and hereditary belongings, apprehending complex behaviours better than traditional differential equations ideology. These types of equations are extensively used in fields like physical sciences [9] biology inspired model [10] and financial analysis [11]. The solutions methodology is quite different and it needs more specialized techniques [12-15].

Theory based on uncertainty play important role for real world modelling now a days. There is several well know ideology to capture the uncertainty when modelling. Few concepts such as interval quantification [16] fuzzy set theory [17] intutionistic fuzzy set [18] theory etc. Fuzzy set consider the degree of belonging ness where as intuitionistic fuzzy set capture both the belongingness and non-belongingness [19, 20]. Apart from the previously mention settings neutrosophic set [21, 22] capture uncertainty than others. The idea of a neutrosophic set is significant because it spreads classical fuzzy set philosophies by letting the depiction of truth, indeterminacy and falsity by making it ideal for manage uncertain, incomplete, and inconsistent info. Contrasting traditional models that might strict boundaries or membership where as neutrosophic sets offer better flexibility and pragmatism in complex decision-making problem like site selection problem [23, 24] mathematical biology [25] Inventory control problem [26]. Transportation problem [27]. Graph theory [28] etc . This makes it mostly powerful in settings in uncertainty modelling.

Differential equation with uncertainty is not new. The most popular differential equation with uncertainty is fuzzy differential equation, which have importance both in theoretical [29, 30] and

History: Received: 12 February 2025; Revised: 22 April 2025; Accepted: 25 April 2025; Published: 14 May 2025
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modelling [31, 32] purposes. Although fuzzy differential equation have different variation like fuzzy fractional differential equation [33, 34] fuzzy delay differential equation [35, 36]. In other hand the neutrosophic differential equation [25, 37-44]. Is taken and solved by few researchers whereas neutrosophic fractional differential equation work is very rare [45]. The details comparative analysis is of published paper based on neutrosophic differential equation show in table 1.

In that context we propose the neutrosphic fractional differential equation by neutrosophic extension principle. The structure of the paper is are follows: Section 1 describes preliminary introduction of related keyworks, Section 2 describes the preliminary ideas. Neutrosophic extension principle is defined in Section 3. The formulation of neutrosophic differential equation is addressed in Section 4. Section 5 stands for the solution strategy of neutrosophic fractional differential equation. Numerical example is illustrated in Section 6. Section 7 stand for conclusion and future research scope.

Table 1.
Comparative study of published neutrosophic differential equation

Sl. No.	Paper details	Approaches used	Type of differential equation	Applications/ Theory
1	Sumathi and Antony Crispin Sweety [37]	Generalized neutrosohic hukuhara differentiability	Second order linear differential equation	Theory
2	Mondal, et al. [38]	(α,β,γ)-cut of neutrosophic function method	First order system of differential equation	Application
3	Sumathi and Priya [39] [39] Sumathi et al.	(α,β,γ)-cut of neutrosophic function method	First order linear homogeneous differential equation	Theory and application both
4	Parikh and Sahni [40] [40] Parikh et al.	Generalized Hukuhara neutrosophic differentiability	First order linear differential equation	Application
5	Rahaman, et al. [41] [41] Rahaman et al.	generalized Neutrosophic derivative	System of linear differential equation	Theory and applications both
6	Acharya, et al. [46] [42] Acharya et al.	Generalized Hukuhara neutrosophic differentiability	First order linear non homogeneous differential equation	Applications
7	Acharya, et al. [25]	Generalized neutrosophic derivative	System of linear non homogenous differential equation	Applications
8	Kamal, et al. [42]	Generalized Hukuhara Differentiability	Second order linear homogeneous	Theory
9	Mera, et al. [43] [45]	Neutrosophic mathematical transform	First order linear homogeneous	Theory
10	Momena, et al. [44]	generalized neutrosophic derivative;	generalized neutrosophic derivative	Application

2. Preliminaries and Basic Concepts

Fuzzy Set: Zadeh [17] A fuzzy set $\tilde{S}$ is well-defined as a set of ordered pair, notationaly as $(r, \mu_{\tilde{S}}(r))$, where $r \in X$, where X is nonempty universal set. The function $\mu_{\tilde{S}}(r): X \to [0,1]$, is called membership function.

Zadeh's extension principle: Zadeh [47] Let J be a crisp set and $\tilde{P}$ be a fuzzy set in J. The function $g: J \to K$ is defined by $k = g(j)$, then the extension principle introduces a fuzzy set $\tilde{R}$ in K as $\tilde{R} = \{(k, \mu_{\tilde{R}}(k) | k = g(j),\ j \in J)\}$ where, $\mu_{\tilde{R}}(k) = \begin{cases} \sup_{j \in g^{-1}(k)} (\mu_{\tilde{P}}(j)), iff\ g^{-1}(k) \neq \phi \\ 0 \quad otherwise \end{cases}$.

Note: Zadeh's Extension Principle permits crisp functions to work on fuzzy sets uncertainty. It encompasses functions by mapping fuzzy inputs functions to fuzzy outputs functions whereas conserving membership grades. The output's membership functions are resolute using the supremum of input memberships function that map to respective output value. This ideology is introductory in fuzzy

arithmetic operations and fuzzy based decision-making. It permits applying in models to capture imprecise data.
Example: Let, $\widetilde{M}_f$ be a fuzzy set given by the membership function as follows:

$$\mu_{\widetilde{M}_f}(j) = \begin{cases} 0 \; if \; j \leq 2 \\ \frac{j-2}{4} \; if \; 2 \leq j < 6 \\ 1 \; if \; j = 6 \\ \frac{10-j}{4} \; if \; 6 < j \leq 10 \\ 0 \; if \; j \geq 10 \end{cases}$$

Let us choose a function $F(j) = 3j + 2$. Using the concept of Zadeh's extension principle, another fuzzy set $F(\widetilde{M}_f)$ can be determined. The membership function of $F(\widetilde{M}_f)$ is obtained as follows:

$$\mu_{F(\widetilde{M}_f)}(k) = \begin{cases} 0 \; if \; k \leq 8 \\ \frac{k-8}{12} \; if \; 8 \leq k < 20 \\ 1 \; if \; k = 20 \\ \frac{32-k}{12} \; if \; 20 < k \leq 32 \\ 0 \; if \; k \geq 32 \end{cases}$$

Note: The above concepts define that how we construct a fuzzy function by considering a parameter or variable as fuzzy in nature. Since the resulting function also obey the fuzzy rules.
Neutrosophic Set: The extension of fuzzy sets is neutrosophic fuzzy sets. Here in the neutrosophic set Smarandache [48] one step forward of the Intuitionistic fuzzy set theory ideology. There exists several form of the said set. One of them is single-valued neutrosophic set. Consider an neutrosophic set $\widetilde{M}_{Ne}$ on universal set U is defined as $\widetilde{M}_{Ne} = \{(T_{Ne}(k), I_{Ne}(k), F_{Ne}(k)) : k \in \mathrm{U}\}$, where $T_{Ne}(k), I_{Ne}(k), F_{Ne}(k) : \mathrm{U} \rightarrow [0,1]$ are considered as the degree of truthness, degree of indeterministic and degree of falsity function respectively for $k \in \mathrm{U}$, such that $0 \leq T_{Ne}(k), I_{Ne}(k), F_{Ne}(k) \leq 3$.

Table 2.
Comparison between Fuzzy, Intutitionistic fuzzy and Neutrosophic fuzzy set idea

Sl. No.	Set	Associated functions	Conditions	Advantage	Disadvantage
1	Fuzzy set	Membership function ($\mu_{\tilde{A}}(x)$)	$0 \leq \mu_{\tilde{A}}(x) \leq 1$	Simple and for vague concepts.	No way to represent contradiction in data.
2	Intutitionistic fuzzy set	Membership and non-membership ($\mu_{\tilde{A}}(x), \vartheta_{\tilde{A}}(x)$)	$0 \leq \mu_{\tilde{A}}(x) + \vartheta_{\tilde{A}}(x) \leq 1$	Captures hesitation	Conditions restricts handling inconsistent or contradictory data.
3	Neutrosophic fuzzy set	Truth, Indeterminacy and Falsity ($T_{\tilde{A}}(x), I_{\tilde{A}}(x), I_{\tilde{A}}(x)$)	$0 \leq T_{\tilde{A}}(x) + I_{\tilde{A}}(x) + I_{\tilde{A}}(x) \leq 1 \; or, 2 \; or, 3$	deal with incomplete, indeterminate, and inconsistent information,	More complex computation and interpretation.

Note: Here in table 2 the comparison between fuzzy set, intuitionistic fuzzy and neutrosophic fuzzy set.

Triangular neutrosophic number: Wang, et al. [22] A Triangular neutrosophic number is taken as $\widetilde{M} = (m_{01}, m_{02}, m_{03};\ m_{11}, m_{12}, m_{13};\ m_{21}, m_{22}, m_{23})$, where the truth membership, indeterminacy and falsity function is fixed as follows:

$$T_{Ne}(k) = \begin{cases} \dfrac{k - m_{01}}{m_{02} - m_{01}} & when\, m_{01} \leq k < m_{02} \\ 1 & when\, k = m_{02} \\ \dfrac{m_{03} - k}{m_{03} - m_{02}} & when\, m_{02} < k \leq m_{03} \\ 0 & otherwise \end{cases}$$

$$I_{Ne}(k) = \begin{cases} \dfrac{m_{12} - k}{m_{12} - m_{11}} & when\, m_{11} \leq k < m_{12} \\ 0 & when\, k = m_{12} \\ \dfrac{k - m_{12}}{m_{13} - m_{12}} & when\, m_{12} < k \leq m_{13} \\ 1 & otherwise \end{cases}$$

$$F_{Ne}(k) = \begin{cases} \dfrac{m_{22} - k}{m_{22} - m_{21}} & when\, m_{21} \leq k < m_{22} \\ 0 & when\, k = m_{22} \\ \dfrac{k - m_{22}}{m_{23} - m_{22}} & when\, m_{22} < k \leq m_{23} \\ 1 & otherwise \end{cases}$$

Where, $0 \leq T_{Ne}(k) + I_{Ne}(k) + F_{Ne}(k) \leq 3$ and $y \in \widetilde{M}$.

Parametric form of triangular neutrosophic number or, (α, β, γ)-cut: The parametric setting of the above number or the (α, β, γ)-cut is

$$[\widetilde{M}]_{\alpha,\beta,\gamma} = \{[M_\alpha^L, M_\alpha^R]; [M_\beta^L, M_\beta^R]; [M_\gamma^L, M_\gamma^R]\}$$

Where
$$\begin{cases} M_\alpha^L = m_{01} + \alpha(m_{02} - m_{01}) \\ M_\alpha^R = m_{03} - \alpha(m_{03} - m_{02}) \\ M_\beta^L = m_{12} - \beta(m_{12} - m_{11}) \\ M_\beta^R = m_{12} + \beta(m_{13} - m_{12}) \\ M_\gamma^L = m_{22} - \beta(m_{22} - m_{21}) \\ M_\gamma^R = m_{22} + \beta(m_{23} - m_{22}) \end{cases}$$

with $0 < \alpha, \beta, \gamma \leq 1$ and $0 < \alpha + \beta + \gamma \leq 3$.

Note: Its is need not necessary that we have to take triangular neutrosophic number. There exist different variation of neutrosophic number such as trapezoidal neutrosophic number, pentagonal neutrosophic number etc.

Mittag-Leffler function: If ${}^cD_b^\delta y(t) = y(t)$, with $y(0) = y_0$ then the solution is $y(t) = y_0 E_\delta(t^\delta)$, where $E_\delta(t^\delta)$ is called Mittag-Leffler function and it expressed as,

$$E_\delta(t^\delta) = \sum_{p=0}^{\infty} \frac{u^p}{\Gamma(p\delta + 1)}, \delta > 0$$

Note: The Mittag-Leffler function have a vital role in fractional calculus. It generalizes the exponential function and arises in the solutions of fractional differential equations. In particularly those problems involving Caputo or Riemann–Liouville derivatives. Dissimilar the exponential, which defines memoryless progressions, the Mittag-Leffler function imprisonments power-law based decay and memory-based effects for making it perfect for modelling real-world spectacles such as **anomalous** diffusion**,** fluid dynamics, biological systems with hereditary properties. It appears when solving the fractional differential equation by using Laplace transform also.

Caputo Derivative: For a function $\nabla(r)$ the Caputo derivative of order $\rho \in (n-1, n)$ where $n \in N$, is defined as,

$$ {}^{c}D_{r}^{\rho}\nabla(r) = \frac{1}{\Gamma(n-\rho)} \int_{0}^{r} \frac{\nabla^{(n)}(y)}{(r-y)^{\rho-n+1}} dy $$

Where $\Gamma(.)$ is the Gamma function and $\nabla^{(n)}$ denoted the nth derivative of ∇.
So, for $0 < \rho < 1$, the above definitions become

$$ {}^{c}D_{r}^{\rho}\nabla(r) = \frac{1}{\Gamma(1-\rho)} \int_{0}^{r} \frac{\nabla^{(1)}(y)}{(r-y)^{\rho}} dy $$

Note: The Caputo derivative is very important in fractional calculus theory because it allows fractional differential equations with initial conditions which may expressed in terms of classical integer-order derivatives. The Caputo derivative particularly suitable for modelling dynamical systems in science, engineering, physics and biological science where initial states are known in classical terms. Moreover, it conserves key properties like linearity and convulsions naturally into Laplace transform methods. It simplifying the analytical solution of fractional differential equations and attractive its real-world applicability in initial value problems.

Caputo derivative for initial value problem: Consider the fractional differential equation of type ${}^{c}D_{y}^{\rho}y(r) = my(r)$ with initial value $y(0) = y_0$, then the solution is written as

$$ y(r) = E_{\rho}(-mr^{\rho}) $$

3. Neutrosophic Extension Principle

Extension Zadeh's extension principle with neutrosophic uncertainty: Let V be a crisp set and $\tilde{B}$ be a neutrosophic set in V. The function $g: V \to Q$ is defined by $q = g(v)$, then the extension principle introduces a neutrosophic set fuzzy set $\tilde{C}$ in Q as $\tilde{C} = \{(q, T_{\tilde{C}}(q), I_{\tilde{C}}(q), F_{\tilde{C}}(q) | q = g(v),\ u \in U)\}$ where,

$$ T_{\tilde{C}}(q) = \begin{cases} \sup\limits_{v \in g^{-1}(q)} \left(T_{\tilde{B}}(v)\right), iff\ g^{-1}(q) \neq \phi \\ 0 \quad otherwise \end{cases} $$

$$ I_{\tilde{C}}(q) = \begin{cases} \inf\limits_{v \in g^{-1}(q)} \left(I_{\tilde{B}}(v)\right), iff\ g^{-1}(q) \neq \phi \\ 1 \quad otherwise \end{cases} $$

and

$$ F_{\tilde{C}}(q) = \begin{cases} \inf\limits_{v \in g^{-1}(q)} \left(F_{\tilde{B}}(v)\right), iff\ g^{-1}(q) \neq \phi \\ 1 \quad otherwise \end{cases} $$

Note: Neutrosophic extension principle ultimately is the extension of Zadey's extension principal. Our main aim is to use the theory for finding the solution of neutrosophic fractional differential equation.

Example: Let, $\tilde{B}_g$ be a neutrosophic set given by the truth, indeterminacy and falsity function as follows:

$$T_{B_g}(x) = \begin{cases} 0 \; if \; x \leq 80 \\ \left(\frac{x-80}{20}\right) if \; 80 \leq x < 100 \\ 1 \; if \; x = 100 \\ \left(\frac{120-x}{20}\right) if \; 100 < x \leq 120 \\ 0 \; if \; x \geq 120 \end{cases}$$

$$I_{B_g}(x) = \begin{cases} 0 \; if \; x \leq 90 \\ \left(\frac{100-x}{10}\right) if \; 90 \leq x < 100 \\ 0 \; if \; x = 100 \\ \left(\frac{x-100}{10}\right) if \; 100 < x \leq 110 \\ 0 \; if \; x \geq 110 \end{cases}$$

and

$$F_{B_g}(x) = \begin{cases} 0 \; if \; x \leq 95 \\ \left(\frac{100-x}{5}\right) if \; 95 \leq x < 100 \\ 0 \; if \; x = 100 \\ \left(\frac{x-100}{5}\right) if \; 100 < x \leq 105 \\ 0 \; if \; x \geq 115 \end{cases}$$

Consider a function $G(x) = x + 50$. Using the concept of Zadeh's extension principle, the neutrosophic set $G(B_g)$ can be determined. The function of $G(B_g)$ is obtained as follows:

$$T_{B_g}(x) = \begin{cases} 0 \; if \; x \leq 130 \\ \left(\frac{x-130}{20}\right) if \; 130 \leq x < 150 \\ 1 \; if \; x = 150 \\ \left(\frac{170-x}{20}\right) if \; 150 < x \leq 170 \\ 0 \; if \; x \geq 170 \end{cases}$$

$$I_{B_g}(x) = \begin{cases} 0 \; if \; x \leq 90 \\ \left(\frac{150-x}{10}\right) if \; 140 \leq x < 140 \\ 0 \; if \; x = 150 \\ \left(\frac{x-150}{10}\right) if \; 150 < x \leq 160 \\ 0 \; if \; x \geq 160 \end{cases}$$

and

Edelweiss Applied Science and Technology
ISSN: 2576-8484
Vol. 9, No. 5: 1393-1405, 2025
DOI: 10.55214/25768484.v9i5.7172

$$F_{B_g}(x) = \begin{cases} 0 \; if \; x \leq 245 \\ \left(\frac{150 - x}{5}\right) \; if \; 145 \leq x < 150 \\ 0 \; if \; x = 150 \\ \left(\frac{x - 150}{5}\right) \; if \; 150 < x \leq 165 \\ 0 \; if \; x \geq 165 \end{cases}$$

Theorem: If $\tilde{Y}(t): [t_0, T] \to F(R)$ is a neutrosophic fuzzy function whose (α, β, γ)- cut are denoted by $(\tilde{Y}(t))_{\alpha,\beta,\gamma} = \{[y_{11}(t,\alpha), y_{12}(t,\alpha)]; [y_{21}(t,\beta), y_{22}(t,\beta)]; [y_{31}(t,\gamma), y_{32}(t,\gamma)]\}$ for $\alpha, \beta, \gamma \in [0,1]$, then

(i) $(\tilde{Y}(t))_{\alpha,\beta,\gamma}$ is nonempty compact subset of R.

(ii) $(\tilde{Y}(t))_{\alpha_1} \subseteq (\tilde{Y}(t))_{\alpha_2}$, $(\tilde{Y}(t))_{\beta_1} \subseteq (\tilde{Y}(t))_{\beta_2}$ and $(\tilde{Y}(t))_{\gamma_1} \subseteq (\tilde{Y}(t))_{\gamma_2}$ for $0 \leq \alpha_1 \leq \alpha_2 \leq 1$, $0 \leq \beta_1 \leq \beta_2 \leq 1$, $0 \leq \gamma_1 \leq \gamma_2 \leq 1$.

4. Neutrosophic Fractional Differential Equation (NFDE)

In this section we introduce neutrosophic fractional differential equation. It is obvious that the crisp fractional differential equation and NFDE is different in nature. The idea of fuzzy differential equation is extended here.

Let us consider the crisp fractional differential equation of the form

$$\begin{cases} {}^{c}D_b^{\delta} y(t) = F(t, k, y(t)) \\ \qquad y(t_0) = y_0 \end{cases} \tag{1}$$

Where $F: [t_0, T] \times R \to R$ is a real valued function, $y_0 \in R$, $k \in R$ is constant and $\delta \in (0,1]$.

The above fractional differential equation (1) is said to be neutrosohic fractional differential equation if one of the following conditions holds:

I. The initial conditions y_0 is neutrosophic fuzzy valued number

II. The coefficient or constant k is neutrosophic fuzzy valued number

III. Both the initial conditions y_0 and coefficient or constant k is neutrosophic fuzzy valued number

Here a question arises that when we consider the above cases ? Basically, for theoretical study any one or all three may considered. But for real life model the one has to take which is best fit for the model considered.

In this study we only consider the first cases i.e., the initial condition is neutrosophic fuzzy valued number. In future study all cases are considered still the idea for solution strategy is quite similar.

Since the initial condition is neutrosophic fuzzy valued so the solutions also neutrosophic fuzzy valued, so we take the whole equations form as follows and treated as neutrosophic fuzzy fractional differential equations:

$$\begin{cases} {}^{c}D_b^{\delta} \tilde{y}(t) = \tilde{F}(t, k, y(t)) \\ \qquad y(t_0) = \tilde{y}_0 \end{cases} \tag{2}$$

Where $\tilde{F}: [t_0, T] \times R_f \to R_f$ is a real valued neutrosophic function, $\tilde{y}_0 \in R_f$, $k \in R$ is constant and $\delta \in (0,1]$.

Note 1: For $\delta = 1$, the equation (1) converted to simple crisp ordinary differential equation and (2) converted to neutrosophic fuzzy differential equation. Also, it should be noted that in solution if we put the integer value of δ then the solution is quite similar by not fully because we have to use some numerical approximation restrictions.

5. Solution of Neutrosophic Fractional Differential Equation Using Extension Principal Method

There are several articles where extension principal is used to solve fuzzy differential equation such as [1-4]. The idea is very popular since the derivative of fuzzy functions concepts or interval arithmetic property is not necessary.

The solution using the extension principle is very much straight forward. Strating from crisp solution, then maximum and minimum consideration with respect to uncertain parameters, then submission the corresponding interval ends the solution procedure is completed. It should be noted that different method may give different solution.

Consider the crisp solution of (2), that is the solution of (1) is as follows:

$$y(t) = \hat{g}(k, y_0, t) \quad (3)$$

In equation (3) the function $\hat{g}$ is crisp. The idea is that fuzzification have to done by extension principal. In the function k, y_0 may be neutrosophic in nature. In this particular paper we take only the initial value as neutrosophic number, so only we have to focus the parameter y_0 for applying Zadeh's extension principle via neutrosophic approach.

and let the (α, β, γ)-cut of the neutrosophic initial value of (2) is

$$(\tilde{y}_0)_{\alpha,\beta,\gamma} = \{[p_1(\alpha), p_2(\alpha)]; [q_1(\beta), q_2(\beta)]; [r_1(\gamma), r_2(\gamma)]\} \quad (4)$$

This above is parametric representation or in interval form.

Theorem: If $(\tilde{y}(t))_{\alpha,\beta,\gamma} = \left\{\hat{g}(k, y_0, t)|_\alpha; \hat{g}(k, y_0, t)|_\beta; \hat{g}(k, y_0, t)|_\gamma\right\}$ is the solution of (2) then

$$\hat{g}(k, y_0, t)|_\alpha = \left[y_{11}(t, \alpha) = min\,\{\hat{g}(k, y_0, t)\}, y_{12}(t, \alpha) = max\,\{\hat{g}(k, y_0, t)\}|\, y_0 \in [p_1(\alpha), p_2(\alpha)]\right]$$

$$\hat{g}(k, y_0, t)|_\beta = \left[y_{21}(t, \beta) = min\,\{\hat{g}(k, y_0, t)\}, y_{22}(t, \beta) = max\,\{\hat{g}(k, y_0, t)\}|\, y_0 \in [q_1(\beta), q_2(\beta)]\right]$$

$$\hat{g}(k, y_0, t)|_\gamma = \left[y_{31}(t, \gamma) = min\,\{\hat{g}(k, y_0, t)\}, y_{32}(t, \gamma) = max\,\{\hat{g}(k, y_0, t)\}|\, y_0 \in [r_1(\gamma), r_2(\gamma)]\right]$$

For $\alpha, \beta, \gamma \in [0,1]$ it is obvious that $y_{11}(t, \alpha) \leq y_{12}(t, \alpha)$, $y_{21}(t, \beta) \leq y_{22}(t, \beta)$ and $y_{31}(t, \gamma) \leq y_{32}(t, \gamma)$.

The above theory shows that how we may find the parametric neutrosophic function with respect to a neutrosophic parameter.

Two cases happen for the $\hat{g}(k, y_0, t)$.

Case 1: $\hat{g}(k, t)$ is increasing with respect to y_0

Then by Zade's extension principle the solutions are written as follows

$$(\tilde{y}(t))_{\alpha,\beta,\gamma} = \left\{\hat{g}(k, y_0, t)|_\alpha; \hat{g}(k, y_0, t)|_\beta; \hat{g}(k, y_0, t)|_\gamma\right\} \quad (5)$$

$$\text{Where} \begin{cases} \hat{g}(k, y_0, t)|_\alpha = [\hat{g}(k, p_1(\alpha), t), \hat{g}(k, p_2(\alpha), t)] \\ \hat{g}(k, y_0, t)|_\beta = [\hat{g}(k, q_1(\beta), t), \hat{g}(k, q_2(\beta), t)] \\ \hat{g}(k, y_0, t)|_\gamma = [\hat{g}(k, r_1(\gamma), t), \hat{g}(k, r_2(\gamma), t)] \end{cases} \quad (6)$$

and another one is

Case 2: $\hat{g}(k, t)$ is decreasing with respect to y_0

Then by Zade's extension principle the solutions are written as follows

$$(\tilde{y}(t))_{\alpha,\beta,\gamma} = \left\{\hat{g}(k, y_0, t)|_\alpha; \hat{g}(k, y_0, t)|_\beta; \hat{g}(k, y_0, t)|_\gamma\right\} \quad (7)$$

$$\text{Where} \begin{cases} \hat{g}(k, y_0, t)|_\alpha = [\hat{g}(k, p_2(\alpha), t), \hat{g}(k, p_1(\alpha), t)] \\ \hat{g}(k, y_0, t)|_\beta = [\hat{g}(k, q_2(\beta), t), \hat{g}(k, q_1(\beta), t)] \\ \hat{g}(k, y_0, t)|_\gamma = [\hat{g}(k, r_2(\gamma), t), \hat{g}(k, r_1(\gamma), t)] \end{cases} \quad (8)$$

Note 2: Same concept is applicable if only coefficient k is neutrosophic number.

Note 3: Four cases happen if k and y_0 both are neutrosophic valued, and the cases are as follows:

Case 1: $\hat{g}(k, t)$ is increasing with respect to k and y_0 both

Case 2: $\hat{g}(k, t)$ is increasing with respect to k but decreasing with respect to y_0

Case 3: $\hat{g}(k, t)$ is decreasing with respect to k and increasing with respect to y_0

Case 4: $\hat{g}(k,t)$ is decreasing with respect to k and y_0 both

It should be noted that every time the result should be checked whether it is obeying the neutrosophic rules or not.

6. Numerical Illustrations

Example 1: Consider the fractional differential equation with neutrosophic initial value

$$\begin{cases} {}^{c}D_{b}^{\rho}\widetilde{\Omega}(t) = \tilde{u}(t) \\ \widetilde{\Omega}_0 = (8,12,16;\ 10,12,14;\ 11,12,15) \end{cases}$$

Solution: The solution associated with crisp fractional differential equation is

$$\Omega(t) = \Omega_0 E_\rho(t^\rho) = \Omega_0\left(1 + \frac{t^\rho}{\Gamma(\rho+1)} + \frac{t^{2\rho}}{\Gamma(2\rho+1)}\right)$$

(we take the approximated form of Mitag-Lafler function up to third term).

Now $\frac{d\Omega(t)}{dt} = \Omega_0\left(1 + \frac{t^\rho}{\Gamma(\rho+1)} + \frac{t^{2\rho}}{\Gamma(2\rho+1)}\right) > 0$,

It shows that the function $\Omega(t)$ is an increasing function with respect to t.

Now the (α,β,γ)-cut of the initial value of $\tilde{u}_0$ is

$(\widetilde{\Omega}_0)_{\alpha,\beta,\gamma} = \{[8+4\alpha, 16-4\alpha]; [12-2\beta, 12+2\beta]; [12-\beta, 12+\beta]\}$

Using equation (5) in section 5, we get the neutrosophic solution

$$\begin{aligned}\left(\widetilde{\Omega}(t)\right)_{\alpha,\beta,\gamma} &= \{[\Omega_\alpha^L(t), \Omega_\alpha^R(t)]; [\Omega_\beta^L(t), \Omega_\beta^R(t)]; [\Omega_\gamma^L(t), \Omega_\gamma^R(t)]\} \\ &= \left\{\left[(8+4\alpha)\left(1 + \frac{t^\rho}{\Gamma(\rho+1)} + \frac{t^{2\rho}}{\Gamma(2\rho+1)}\right), (16-4\alpha)\left(1 + \frac{t^\rho}{\Gamma(\rho+1)} \right.\right.\right. \\ &\left.\left. + \frac{t^{2\rho}}{\Gamma(2\rho+1)}\right)\right]; \left[(12-2\beta)\left(1 + \frac{t^\rho}{\Gamma(\rho+1)} + \frac{t^{2\rho}}{\Gamma(2\rho+1)}\right), (12+2\beta)\left(1 + \frac{t^\rho}{\Gamma(\rho+1)}\right.\right. \\ &\left.\left. + \frac{t^{2\rho}}{\Gamma(2\rho+1)}\right)\right]; \left[(12-\gamma)\left(1 + \frac{t^\rho}{\Gamma(\rho+1)} + \frac{t^{2\rho}}{\Gamma(2\rho+1)}\right), (12+\gamma)\left(1 + \frac{t^\rho}{\Gamma(\rho+1)}\right.\right. \\ &\left.\left.\left. + \frac{t^{2\rho}}{\Gamma(2\rho+1)}\right)\right]\right\}\end{aligned}$$

Now the pictorial representation of the solution for $\rho = 0.75$ and $t \in [0,100]$ is as follows:

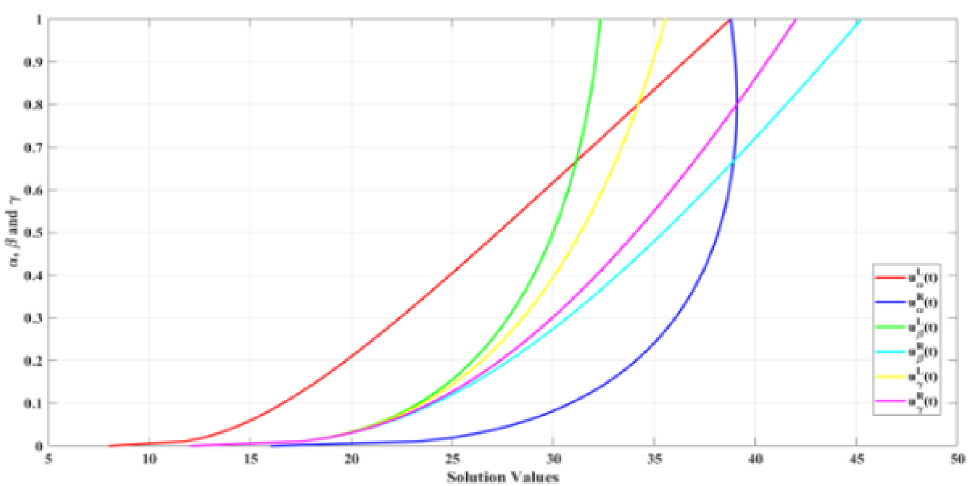

Figure 1.
Solution for $\rho = 0.75$ and $t \in [0,100]$
Note: Clearly, we that the solution obeys the conditions

$$\Omega_{\alpha}^{L}(t) \leq \Omega_{\alpha}^{R}(t), \Omega_{\beta}^{L}(t) \leq \Omega_{\beta}^{R}(t), \Omega_{\gamma}^{L}(t) \leq \Omega_{\gamma}^{R}(t)$$

for particular $\rho = 0.75$ and $t \in [0,100]$, $\alpha, \beta, \gamma \in [0,1]$, therefore it also is a neutrosophic solution.

Example 2: Consider the fractional differential equation with neutrosophic initial value

$$\begin{cases} {}^{C}D_{b}^{\rho}\widetilde{\Delta}(t) = -\widetilde{\Delta}(t) \\ \widetilde{\Delta}_0 = (30,50,70;\ 40,50,60;\ 45,50,55) \end{cases}$$

Solution: The crisp solution is $\Delta(t) = \Delta_0 E_{\rho}(-t^{\rho}) = \Delta_0\left(-1 - \frac{t^{\rho}}{\Gamma(\rho+1)} - \frac{t^{2\rho}}{\Gamma(2\rho+1)}\right)$ (approximated up to third term).

Now $\frac{d\Delta(t)}{dt} = \Delta_0\left(-1 - \frac{t^{\rho}}{\Gamma(\rho+1)} - \frac{t^{2\rho}}{\Gamma(2\rho+1)}\right) < 0$,

which is an increasing function with respect to t.

The (α, β, γ)-cut of the initial value of $\widetilde{\Delta}_0$ is

$(\widetilde{\Delta}_0)_{\alpha,\beta,\gamma} = \{[30 + 20\alpha, 70 - 20\alpha]; [50 - 10\beta, 50 + 10\beta]; [50 - 5\beta, 50 + 5\beta]\}$

Using equation (6) from section 5 we get

$$\begin{aligned} \left(\widetilde{\Delta}(t)\right)_{\alpha,\beta,\gamma} &= \left\{[\Delta_{\alpha}^{L}(t), \Delta_{\alpha}^{R}(t)]; \left[\Delta_{\beta}^{L}(t), \Delta_{\beta}^{R}(t)\right]; \left[\Delta_{\gamma}^{L}(t), \Delta_{\gamma}^{R}(t)\right]\right\} \\ &= \left\{\left[(70 - 20\alpha)\left(-1 - \frac{t^{\rho}}{\Gamma(\rho+1)} - \frac{t^{2\rho}}{\Gamma(2\rho+1)}\right), (30 + 20\alpha)\left(-1 - \frac{t^{\rho}}{\Gamma(\rho+1)} \right.\right.\right. \\ &\left.\left. - \frac{t^{2\rho}}{\Gamma(2\rho+1)}\right)\right]; \left[(50 + 10\beta)\left(-1 - \frac{t^{\rho}}{\Gamma(\rho+1)} - \frac{t^{2\rho}}{\Gamma(2\rho+1)}\right), (50 - 10\beta)\left(-1 \right.\right. \\ &\left.\left. - \frac{t^{\rho}}{\Gamma(\rho+1)} - \frac{t^{2\rho}}{\Gamma(2\rho+1)}\right)\right]; \left[(50 + 5\gamma)\left(-1 - \frac{t^{\rho}}{\Gamma(\rho+1)} - \frac{t^{2\rho}}{\Gamma(2\rho+1)}\right), (50 - 5\gamma)\left(-1 \right.\right. \\ &\left.\left.\left. - \frac{t^{\rho}}{\Gamma(\rho+1)} - \frac{t^{2\rho}}{\Gamma(2\rho+1)}\right)\right]\right\} \end{aligned}$$

Now the pictorial representation of the solution for $\rho = 0.25$ and $t \in [0,100]$ is as follows:

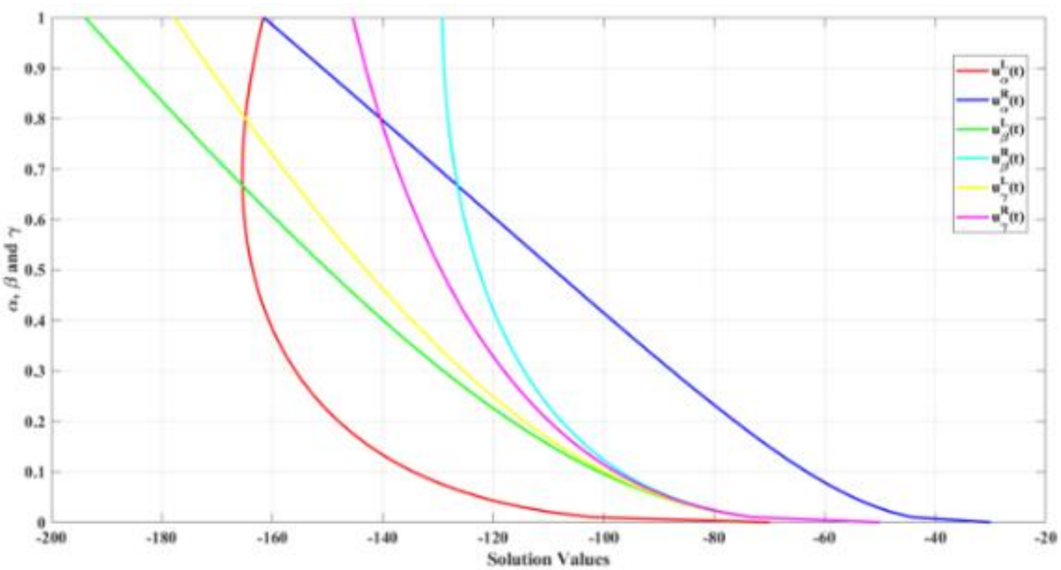

Figure 2.
Solution for $\rho = 0.25$ and $t \in [0,100]$
Note: Clearly, we that the solution obeys the conditions

$$\Delta_{\alpha}^{L}(t) \leq \Delta_{\alpha}^{R}(t), \Delta_{\beta}^{L}(t) \leq \Delta_{\beta}^{R}(t), \Delta_{\gamma}^{L}(t) \leq \Delta_{\gamma}^{R}(t)$$

for particular $\rho = 0.25$ and $t \in [0,100]$, $\alpha, \beta, \gamma \in [0,1]$, therefore it also is a neutrosophic solution.

7. Conclusion and Future Research Scope

In this paper, we have illustrated neutrosophic extension principal method for efficiently solving the neutrosophic fuzzy fractional differential equations (NFFDEs). By participating the extension principal method into the framework of fractional calculus ideology in neutrosophic fuzzy environments a complex system formed. The advanced method extends classical solution strategy to handle neutrosophic fuzzy initial conditions. Several illustrative examples demonstrated the viability, reliability, and flexibility of the proposed method. Overall, the proposed methods provide a systematic, reliable, and adaptable tool for dealing with fractional calculus systems influenced by multifaceted uncertainties like neutrosophic sets.

There are various aspects for future research extension based on the present work. One of the proposals is to extend the methodology into system of fractional differential equations with neutrosophic uncertainty. Another development done for nonlinear systems numerical algorithm findings rather than the analytical solutions which may difficult sometimes to obtained. By considering several fractional derivatives like the Caputo–Fabrizio or Atangana–Baleanu derivatives with the neutrosophic fuzzy settings which could also improve the model's capability to capture different types of memory effects. The uncertainty parameters also change with respect to decision makers need, the extension of neutrosophic sets such as cylindrical neutrosophic sets may considered. The core applications may be found from the field like engineering science, mathematical biology and economics for adopting the fractional calculus theory with uncertainty. Future research also focusses on by integrating optimization methods in the solution methdology, which allowing for parameter identification and system optimization in various complex neutrosophic fractional modelling.

Transparency:

The author confirms that the manuscript is an honest, accurate, and transparent account of the study; that no vital features of the study have been omitted; and that any discrepancies from the study as planned have been explained. This study followed all ethical practices during writing.

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References

[1] K. L. Cooke, "Differential—difference equations," in *International Symposium on Nonlinear Differential Equations and Nonlinear Mechanics*, 1963, pp. 155-171.

[2] C. C. Ross, "About differential equations." New York: Springer, 2004, pp. 1-25.

[3] M. Braun and M. Golubitsky, *Differential equations and their applications*. New York: Springer-Verlag, 1983.

[4] K. Sobczyk, *Stochastic differential equations: With applications to physics and engineering*. Springer Science & Business Media. https://doi.org/10.1007/978-94-011-3712-6, 2013.

[5] A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo, *Theory and applications of fractional differential equations*. Elsevier, 2006.

[6] I. Podlubny, *Fractional differential equations: an introduction to fractional derivatives, fractional differential equations, to methods of their solution and some of their applications*. Elsevier, 1998.

[7] Y. Zhou, *Basic theory of fractional differential equations*. World Scientific, 2023.

[8] V. Lakshmikantham and A. S. Vatsala, "Basic theory of fractional differential equations," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 69, no. 8, pp. 2677-2682, 2008. https://doi.org/10.1016/j.na.2007.08.042

[9] V. P. Dubey, J. Singh, A. M. Alshehri, S. Dubey, and D. Kumar, "Analysis and fractal dynamics of local fractional partial differential equations occurring in physical sciences," *Journal of Computational and Nonlinear Dynamics*, vol. 18, no. 3, p. 031001, 2023. https://doi.org/10.1115/1.4056360

[10] Z. Sabir, S. A. Bhat, H. A. Wahab, M. E. Camargo, G. Abildinova, and Z. Zulpykhar, "A bio inspired learning scheme for the fractional order kidney function model with neural networks," *Chaos, Solitons & Fractals*, vol. 180, p. 114562, 2024. http://dx.doi.org/10.1016/j.chaos.2024.114562
[11] I. Ali and S. U. Khan, "A dynamic competition analysis of stochastic fractional differential equation arising in finance via pseudospectral method," *Mathematics*, vol. 11, no. 6, p. 1328, 2023. https://doi.org/10.3390/math11061328
[12] A. Arikoglu and I. Ozkol, "Solution of fractional differential equations by using differential transform method," *Chaos, Solitons & Fractals*, vol. 34, no. 5, pp. 1473-1481, 2007. https://doi.org/10.1016/j.chaos.2006.09.004
[13] C. Li and F. Zeng, "Finite difference methods for fractional differential equations," *International Journal of Bifurcation and Chaos*, vol. 22, no. 04, p. 1230014, 2012. http://dx.doi.org/10.1142/S0218127412300145
[14] A. Arikoglu and I. Ozkol, "Solution of fractional integro-differential equations by using fractional differential transform method," *Chaos, Solitons & Fractals*, vol. 40, no. 2, pp. 521-529, 2009. https://doi.org/10.1016/j.chaos.2007.08.001
[15] A. Akgül, M. Inc, E. Karatas, and D. Baleanu, "Numerical solutions of fractional differential equations of Lane-Emden type by an accurate technique," *Advances in Difference Equations*, vol. 2015, pp. 1-12, 2015. https://doi.org/10.1186/s13662-015-0558-8
[16] S. Ferson, V. Kreinovich, J. Hajagos, W. L. Oberkampf, and L. Ginzburg, "Experimental uncertainty estimation and statistics for data having interval uncertainty," Sandia National Laboratories (SNL), Albuquerque, NM, and Livermore, CA. https://doi.org/10.2172/910198, 2007.
[17] L. A. Zadeh, "Fuzzy sets," *Information and Control*, vol. 8, no. 3, pp. 338-353, 1965. https://doi.org/10.1016/S0019-9958(65)90241-X
[18] K. Atanassov, "Intuitionistic fuzzy sets," *International Journal Bioautomation*, vol. 20, p. 1, 2016.
[19] V. Khatibi and G. A. Montazer, "Intuitionistic fuzzy set vs. fuzzy set application in medical pattern recognition," *Artificial Intelligence in Medicine*, vol. 47, no. 1, pp. 43-52, 2009. https://doi.org/10.1016/j.artmed.2009.03.002
[20] K. Guo, "Knowledge measure for Atanassov's intuitionistic fuzzy sets," *IEEE Transactions on Fuzzy Systems*, vol. 24, no. 5, pp. 1072-1078, 2015. https://doi.org/10.1109/TFUZZ.2015.2501434
[21] A. Salama and S. Alblowi, "Neutrosophic set and neutrosophic topological spaces," 2012.
[22] H. Wang, F. Smarandache, Y. Zhang, and R. Sunderraman, *Single valued neutrosophic sets*. Infinite study, 2010.
[23] T. Basuri, K. H. Gazi, P. Bhaduri, S. G. Das, and S. P. Mondal, "Decision-analytics-based sustainable location problem-neutrosophic critic-copras assessment model," *Management Science Advances*, vol. 2, no. 1, pp. 19-58, 2025. https://doi.org/10.31181/msa2120257
[24] A. Biswas, K. H. Gazi, P. Bhaduri, and S. P. Mondal, "Neutrosophic fuzzy decision-making framework for site selection," *Journal of Decision Analytics and Intelligent Computing*, vol. 4, no. 1, pp. 187-215, 2024. https://doi.org/10.31181/jdaic10004122024b
[25] A. Acharya *et al.*, "Stability analysis of diabetes mellitus model in neutrosophic fuzzy environment," *Franklin Open*, vol. 8, p. 100144, 2024.
[26] R. Haque, M. Rahaman, S. Alam, P. K. Behera, and S. P. Mondal, "Generalized neutrosophic Laplace transform and its application in an EOQ model with price and deterioration-dependent demand," *OPSEARCH*, pp. 1-33, 2024. https://doi.org/10.1007/s12597-024-00803-y
[27] M. Karak *et al.*, "A solution technique of transportation problem in neutrosophic environment," *Neutrosophic Systems with Applications*, vol. 3, pp. 17-34, 2023. https://doi.org/10.61356/j.nswa.2023.5
[28] T. S. Lakhwani, K. Mohanta, A. Dey, S. P. Mondal, and A. Pal, "Some operations on dombi neutrosophic graph," *Journal of Ambient Intelligence and Humanized Computing*, pp. 1-19, 2022. https://doi.org/10.1007/s12652-021-02909-3
[29] O. Kaleva, "Fuzzy differential equations," *Fuzzy Sets and Systems*, vol. 24, no. 3, pp. 301-317, 1987. https://doi.org/10.1016/0165-0114(87)90029-7
[30] J. J. Buckley and T. Feuring, "Fuzzy differential equations," *Fuzzy sets and Systems*, vol. 110, no. 1, pp. 43-54, 2000. https://doi.org/10.1016/S0165-0114(98)00141-9
[31] S. Chakraverty, S. Tapaswini, and D. Behera, *Fuzzy differential equations and applications for engineers and scientists*. CRC Press, 2016.
[32] A. Bandyopadhyay and S. Kar, "System of type-2 fuzzy differential equations and its applications," *Neural Computing and Applications*, vol. 31, no. 9, pp. 5563-5593, 2019. https://doi.org/10.1007/s00521-018-3380-x
[33] S. Salahshour, T. Allahviranloo, and S. Abbasbandy, "Solving fuzzy fractional differential equations by fuzzy Laplace transforms," *Communications in Nonlinear Science and Numerical Simulation*, vol. 17, no. 3, pp. 1372-1381, 2012. https://doi.org/10.1016/j.cnsns.2011.07.005
[34] T. Allahviranloo, "Fuzzy fractional differential operators and equations," *Studies in fuzziness and soft computing*, vol. 397, 2021. https://doi.org/10.1007/978-3-030-51272-9
[35] N. T. K. Son, H. V. Long, and N. P. Dong, "Fuzzy delay differential equations under granular differentiability with applications," *Computational and Applied Mathematics*, vol. 38, no. 3, p. 107, 2019. https://doi.org/10.1007/s40314-019-0881-x

[36] A. F. Jameel, N. R. Anakira, A. Alomari, M. Al-Mahameed, and A. Saaban, "A new approximate solution of the fuzzy delay differential equations," *International Journal of Mathematical Modelling and Numerical Optimisation*, vol. 9, no. 3, pp. 221-240, 2019. https://doi.org/10.1504/IJMMNO.2019.100476

[37] I. Sumathi and C. Antony Crispin Sweety, "New approach on differential equation via trapezoidal neutrosophic number," *Complex & Intelligent Systems*, vol. 5, pp. 417-424, 2019. https://doi.org/10.1007/s40747-019-00117-3

[38] D. Mondal, S. Tudu, G. C. Roy, and T. K. Roy, "A model describing the neutrosophic differential equation and its application on mine safety," *Neutrosophic Sets and Systems*, vol. 46, pp. 386-401, 2021. https://doi.org/10.5281/zenodo.5486247

[39] I. Sumathi and V. M. Priya, *A new perspective on neutrosophic differential equation*. Infinite Study, 2018.

[40] M. Parikh and M. Sahni, "Solution of logistic differential equation in an uncertain environment using neutrosophic numbers," *Journal of Interdisciplinary Mathematics*, vol. 27, pp. 145-169, 2024. https://doi.org/10.47974/JIM-1795

[41] M. Rahaman, S. P. Mondal, S. Ahmad, K. H. Gazi, and A. Ghosh, "Study of the system of uncertain linear differential equations under neutrosophic sense of uncertainty," *Spectrum of Engineering and Management Sciences*, vol. 3, no. 1, pp. 93-109, 2025. https://doi.org/10.31181/sems31202536r

[42] B. Kamal, H. E. Khalid, A. Salama, and G. I. El-Baghdady, "Finite difference method for neutrosophic fuzzy second order differential equation under generalized hukuhara differentiability," *Neutrosophic Sets and Systems*, vol. 58, no. 1, p. 19, 2023. https://doi.org/10.5281/zenodo.8404478

[43] A. J. Mera, H. A. Hadi, and S. M. Jabbar, "Solving of first order initial value problem using fuzzy Kamal transform in neutrosophic environment," *International Journal of Neutrosophic Science*, vol. 25, no. 3, 2025.

[44] A. F. Momena, R. Haque, M. Rahaman, S. Salahshour, and S. P. Mondal, "The existence and uniqueness conditions for solving neutrosophic differential equations and its consequence on optimal order quantity strategy," *Logistics*, vol. 8, no. 1, p. 18, 2024. https://doi.org/10.3390/logistics8010018

[45] N. T. K. Son, N. P. Dong, H. V. Long, L. H. Son, and A. Khastan, "Linear quadratic regulator problem governed by granular neutrosophic fractional differential equations," *ISA transactions*, vol. 97, pp. 296-316, 2020. https://doi.org/10.1016/j.isatra.2019.11.016

[46] A. Acharya *et al.*, "A neutrosophic differential equation approach for modelling glucose distribution in the bloodstream using neutrosophic sets," *Decision Analytics Journal*, vol. 8, p. 100264, 2023. https://doi.org/10.1016/j.dajour.2023.100264

[47] L. A. Zadeh, "The concept of a linguistic variable and its application to approximate reasoning—I," *Information sciences*, vol. 8, no. 3, pp. 199-249, 1975. https://doi.org/10.1016/0020-0255(75)90036-5

[48] F. Smarandache, "Neutrosophy: Neutrosophic probability, set, and logic: analytic synthesis & synthetic analysis," 1998.