Edelweiss Applied Science and Technology
ISSN: 2576-8484
Vol. 9, No. 7, 162-175
2025
Publisher: Learning Gate
DOI: 10.55214/25768484.v9i7.8557

Improving finite-difference schemes in the context of Nero-hydrodynamic calculations: A conservative approach to solving transfer equations

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Abstract: The purpose of this study was to address critical limitations in existing finite-difference schemes for solving convective transfer equations in aero-hydrodynamic calculations, particularly the non-conservative behavior of the widely used Courant-Isaacson-Rees scheme under specific velocity distributions. The methodology involved a comprehensive analysis of finite-difference schemes using the mass conservation equation for transported substances in compressible media. Test calculations were performed on one-dimensional and two-dimensional problems with varying velocity fields, including cases with velocity sign changes and zero-velocity zones. The proposed scheme uses max and min functions to blend positive and negative velocity components. This approach maintains conservation. The findings demonstrate that the traditional Courant scheme loses conservation when velocity signs change, particularly at stagnation points, leading to mass loss or artificial mass generation. In contrast, the new conservative finite-difference scheme maintains exact mass conservation, stability, and symmetry. It performs well with all tested velocity distributions, even in challenging cases where traditional schemes struggle. In conclusion, the developed scheme eliminates non-conservative behavior that affected existing methods, ensuring accurate representation of physical processes in hydrodynamic calculations. The practical implications include the ability to use larger time steps in numerical calculations while maintaining accuracy and stability, making it particularly valuable for complex aero-hydrodynamic simulations involving flow separation, stagnation points, and variable velocity fields.

Keywords: *Finite differences, Numerical solution, Partial derivative, Scheme viscosity, Stability, Substance transfer.*

1. Introduction

This research aimed to develop a finite-difference scheme for hydrodynamic calculations, avoiding the drawbacks of the Courant scheme. Our goal was to create a scheme that is stable, conservative, transferable, and symmetrical for any velocity distribution. In hydrodynamics, researchers work with fundamental equations, including the Navier-Stokes equations, the equation of conservation of thermal energy, and the equation of conservation of mass of the transported substance [1-9].

The Navier-Stokes equations are central to hydrodynamics, describing the motion of fluid substances. Derived from the conservation of momentum, these equations apply to both relativistic and non-relativistic fluids. They show how the energy-momentum tensor and particle number current are conserved, depending on local hydrodynamic variables like temperature, fluid velocity, and chemical potential. To ensure the stability and causality of these equations, fitting non-equilibrium definitions for the hydrodynamic variables must be used [1, 2].

History: Received: 14 April 2025; Revised: 10 June 2025; Accepted: 13 June 2025; Published: 3 July 2025
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Recent advances in hydrodynamic modeling have gone beyond the traditional Navier-Stokes framework to address limitations in modeling complex flows. For instance, high-order accurate entropy-stable adaptive moving mesh finite difference schemes have been developed for special relativistic hydrodynamics, demonstrating superior performance in handling complex flow phenomena [10]. The Burnett hydrodynamics provides a higher-order extension of the Navier-Stokes equations, derived through various theoretical approaches like the Chapman-Enskog and Grad's methods [3, 4].

Contemporary research has also explored innovative numerical methods for transport equations. Deep neural networks have emerged as powerful tools for approximating solutions of linear transport equations, achieving high accuracy and efficiency, with theoretical confirmation of convergence to analytical solutions [11]. Moreover, numerical schemes for 2D transport equations have been developed, which are unconditionally stable, achieving second-order accuracy and preserving fundamental physical properties [12].

The Vlasov-Navier-Stokes system models the interaction between particles and a fluid, especially in high-friction regimes where the particles are much lighter than the fluid. This leads to the development of Transport-Navier-Stokes systems, which consider non-constant fluid density. The hydrodynamic limit of this system is crucial for understanding how particle-fluid interactions evolve over time [5].

These equations all contain terms that account for the conservative transfer of momentum, energy, or substance. Researchers refer to these terms as inertial, quadratic, or convective terms [1-9]. Essentially, these terms represent the components of the differential operator: the divergence (or convergence) of the flow of momentum, thermal energy, or the mass of the transported impurity.

To find approximate solutions to these equations, researchers use finite difference methods, as well as finite element methods [13-18]. The specific requirements of the problem determine whether to use a finite difference (FD) or finite element (FE) method. When dealing with complex geometries and boundary conditions, engineers tend to prefer FE due to its flexibility and accuracy, whereas they often opt for FD for simpler, regular domains. Both methods are essential in computational mathematics, particularly when solving differential equations [17].

New work on finite difference schemes focuses on improving conservation properties and accuracy. A new method has been proposed for the finite difference approximation of the advective term in heat or solute transport equations. By averaging the advective term at cell boundaries instead of at cell centers [19] this method ensures mass conservation. It consistently produces better results than traditional centered schemes while always preserving mass conservation.

Finite differences are used to form algebraic analogs of differential terms. The resulting algebraic equations are then solved using algebraic methods. In computer hydromechanics, researchers face many challenges when solving these algebraic equations, which relate to the differential equations of compressible medium mechanics [20-22]. A key problem is providing a clear description and precise definition of the terms that explain how momentum, thermal energy, or substances move in a conservative manner.

Advanced computational methods have been developed to address high-dimensional problems in transport equations. AI tools have shown promise in solving complex neutron transport equations, with deep neural networks able to predict multiplication factors with engineering-standard accuracy [23, 24]. These approaches offer new alternatives for engineering design and practical calculations.

Many schemes have been devised for constructing an adequate algebraic analog of the differential terms responsible for transfer. These schemes are easily recognizable and often bear the names of the scientists who developed or studied them. Some schemes, when studied in detail, appear like clever tricks performed on numbers. For example, the well-studied "leapfrog" scheme [25] can be seen as a clever trick by introducing slight variability in the transfer speed.

The Courant-Isaacson-Rees scheme (hereafter referred to as the Courant scheme), also known as the "upstream scheme" [26] is the most widely used scheme in finite difference methods. Researchers have created advanced finite difference schemes for compressible two-medium flows, tackling challenges in multiphase flow calculations [27].

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Using explicit calculation methods [13, 27] and setting fixed time steps [25, 26] allows researchers to obtain a stable, conservative transfer scheme. It is also important that this scheme is almost always invariant (symmetric) with respect to the sign of velocities.

However, the Courant scheme may not remain conservative for some velocity distributions. In non-stationary cases, it can also lose its symmetry and fit with real physical processes. To address these limitations, we analyze finite-difference schemes and conduct computational experiments to compare results from various methods.

2. Research Results

To conduct the study, we had to select an equation. We could have chosen the Navier-Stokes equation, the continuity equation, the equation of conservation of thermal energy, or the equation for conservation of additional mass *S*. The equation of conservation of momentum and the equation of conservation of thermal energy are hard to analyze because they contain not only terms responsible for the transfer and conservation of momentum and heat but also other components. The continuity equation is even more difficult to understand than the equation of conservation of mass. In these equations, the terms responsible for the transfer of momentum or mass are not fundamentally different. All conclusions from the finite-difference scheme for equations (1) and (2) are correct and effective for the Navier-Stokes equation, the thermal energy conservation equation, and the continuity equation. We chose Equation (1) for our research. Equation (1) is the equation of conservation of mass of additional mass *S* transported by a compressible medium [7-9]. We consider the velocity field $\vec{V}$ of the compressible medium to be set as:

$$\frac{\partial S}{\partial t} + div\vec{V}S = 0. \quad (1)$$

Equation (1) can be written in expanded form (2):

$$\frac{\partial S}{\partial t} + \vec{V}gradS + Sdiv\vec{V} = 0 \quad (2)$$

Equations (1) and (2), written for the one-dimensional case ((3) and (4)), will be studied. All conclusions obtained for a one-dimensional problem remain valid for both plane and spatial forms of equations (1) and (2):

$$\frac{\partial S}{\partial t} + \frac{\partial SV}{\partial x} = 0 \quad (3)$$

$$\frac{\partial S}{\partial t} + V\frac{\partial S}{\partial x} + S\frac{\partial V}{\partial x} = 0 \quad (4)$$

Explicit finite-difference analogs of equations (1) and (2) will be studied. Explicit schemes are schemes in which, in future and defined time layers, the sought-for variable is recorded in only one spatial node [13-15, 25, 28]. Thanks to this, solving algebraic equations becomes simple [13, 25].

For the test problem, we choose a solution domain in the form of a straight-line segment. Let us define a velocity field with a modulus equal to one over almost the entire segment, and which changes sign to the opposite at the central point of the segment (see Fig. 1).

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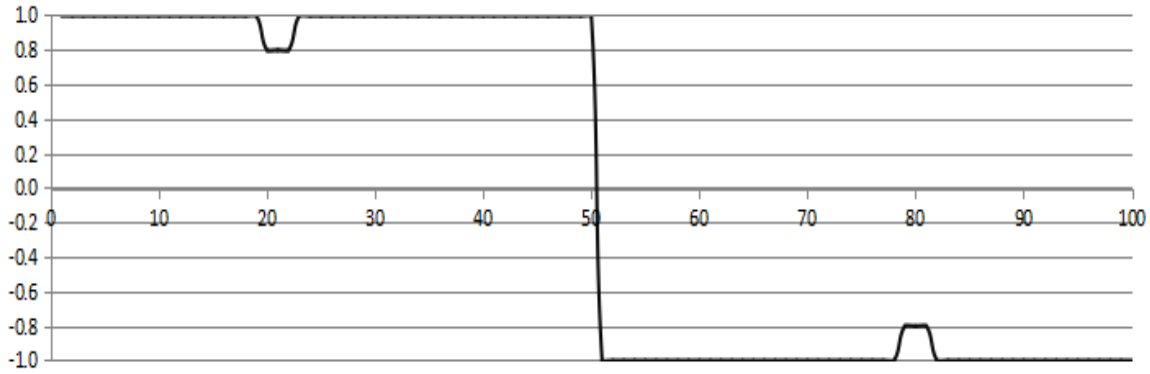

Figure 1.
Velocity distribution along the segment for the first test problem. Initial velocity distribution on the calculated segment of 100 design nodes.

Figure 1 shows two domains where the modulus of velocity is less than one. We compared solutions obtained using different schemes at variable velocities of additional mass motion. The velocity diagram is symmetrical relative to the solution domain center to evaluate the symmetry of the resulting solution with symmetrical initial data. At the initial time point, an additional mass S of 10 units is specified at nodes 10 and 90. The Courant-Isaacson-Rees scheme is used to compare the performance of the proposed scheme. This scheme is widely used in practical hydrodynamics problem calculations due to its stability, conservativeness, symmetry, and adequacy. Note that there are other schemes for calculating transfer equations, but many are complex or unstable, presenting only scientific interest [25]. The Courant-Isaacson-Rees scheme is the most widely used. It has different names, such as "the scheme of directed differences," "upstream scheme," or "corner scheme" [20, 21] but its essence and structure remain the same. The scheme is written as follows:

$$\begin{cases} \frac{S_i^{t+1}-S_i^t}{\Delta t}+\frac{V_i S_i^t-V_{i-1}S_{i-1}^t}{\Delta x}=0, V_i>0 \\ \frac{S_i^{t+1}-S_i^t}{\Delta t}+\frac{V_{i+1}S_{i+1}^t-V_i S_i^t}{\Delta x}=0, V_i<0 \end{cases}, \quad (5)$$

where Δt and Δx are the time step and the step along the length of the segment of the solution domain.

Nevertheless, why was the notation in the form (2) given above? There are a number of reasons for this.

On its basis and in complete analogy with structure (5), it is possible to write a solution-stable, transport, often conservative, and symmetric scheme for calculating additional mass transfer for velocities of the same sign:

$$\begin{cases} \frac{S_i^{t+1}-S_i^t}{\Delta t}+V_i\frac{S_i^t-S_{i-1}^t}{\Delta x}+S_i^t\frac{V_{i+1}-V_i}{\Delta x}=0, V_i>0 \\ \frac{S_i^{t+1}-S_i^t}{\Delta t}+V_i\frac{S_{i+1}^t-S_i^t}{\Delta x}+S_i^t\frac{V_i-V_{i-1}}{\Delta x}=0, V_i<0 \end{cases} \quad (6)$$

Scheme (6) has the only advantage over scheme (5). In each calculation of S_i^{t+1} for point i, it is as if "rests" on three points "S_{i-1}^t","S_i^t","S_{i+1}^t". Scheme (5) rests on only two points in each of the two branches of the possible calculation.

However, almost no one ever uses scheme (6) in calculations. Researchers dealing with hydrodynamic aspects avoid it. They have studied it very little. When using it, calculations often end with an emergency stop. The article will answer why scheme (6) is almost never used in calculations. Using scheme (6) allowed us to create a new scheme (7). This new scheme is better for consumers than scheme (5) and avoids the major issues found in scheme (6) [29, 30]:

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$$\begin{cases} \frac{S_i^{t+1}-S_i^t}{\Delta t} + U_i \frac{S_i^t - S_{i-1}^t}{\Delta x} + S_i^t \frac{U_{i+1}-U_i}{\Delta x} + W_i \frac{S_{i+1}^t - S_i^t}{\Delta x} + S_i^t \frac{W_i - W_{i-1}}{\Delta x} = 0 \\ U_i = \max(V_i, 0), W_i = \min(V_i, 0) \end{cases} \quad (7)$$

We expect the extra mass S to move toward the center point. Here, the velocity will change direction.

The calculation is conducted with the maximum time step for explicit schemes; that is, with the Courant number K_0 equal to one:

$$K_o = \frac{\Delta t}{\Delta x} V \quad (8)$$

The Courant scheme and its version for equation (2) provide a stable and conservative transfer result. Then, at the central point, the calculated content of additional mass for the Courant scheme vanishes to zero. The scheme based on Courant's idea (upstream difference scheme) for the expanded form (2) increases the extra mass concentration at the point where the velocity changes sign (Fig. 2).

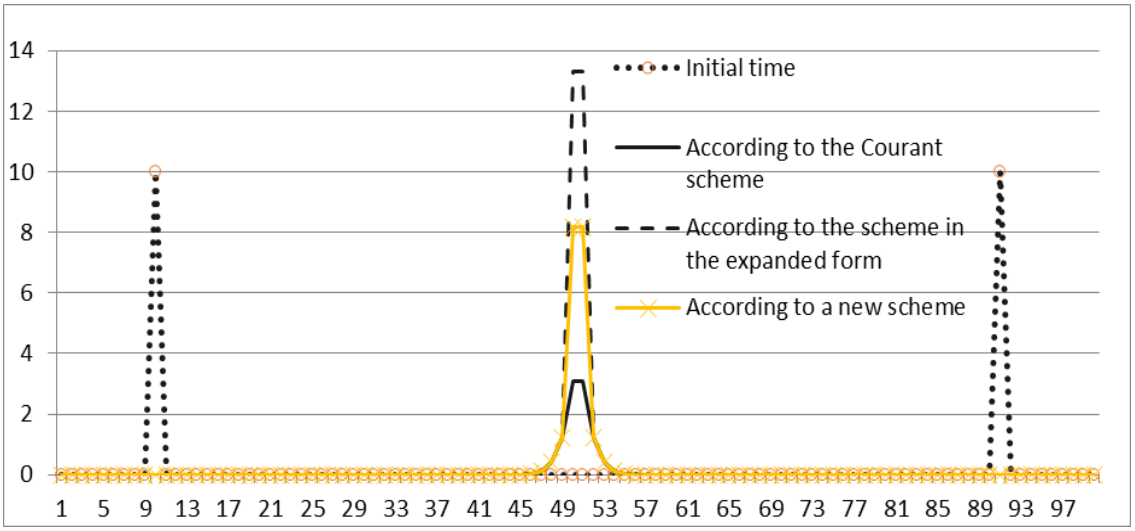

Figure 2.
Solution to the problem of additional mass transfer according to schemes (5), (6), (7). Calculation results for the 41st time step (Courant's parameter is equal to one).

Let us recall that at the initial time point, the concentration of substance *S* was 20 units. At the 41st time step, the Courant scheme gave the amount of substance 9.76. Scheme (6) at the 41st step of calculation gave 30.24. Only scheme (7) ensured the conservativeness of the calculation, and there were 20 units of substance left on the calculated segment, as it was at the very beginning.

It becomes clear why researchers in hydrodynamics avoid using scheme (6) in practical calculations. An infinite increase in the estimated amount of substance will certainly lead to an emergency stop of calculations.

Let us try to slightly change the problem posed. Let there be a section in the central zone of the calculation domain in which there is no motion and velocity is zero. Figure 3 shows the results of this calculation.

It is interesting that the finite-difference scheme (6) has recovered its conservatism. Courant's scheme (5) is still not conservative.

The new scheme and scheme (6) worked perfectly (conservatively, transfer, symmetrically) in the case of zero velocities in the center of the calculated zone.

The point is that the Courant scheme uses strict inequalities. Let us try to write down the Courant scheme for non-strict inequalities (9).

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$$\begin{cases} \frac{S_i^{t+1}-S_i^t}{\Delta t} + V_i \frac{S_i^t - S_{i-1}^t}{\Delta x} + S_i^t \frac{V_{i+1}-V_i}{\Delta x} = 0, V_i \geqslant 0 \\ \frac{S_i^{t+1}-S_i^t}{\Delta t} + V_i \frac{S_{i+1}^t - S_i^t}{\Delta x} + S_i^t \frac{V_i - V_{i-1}}{\Delta x} = 0, V_i \leqslant 0 \end{cases} \quad (9)$$

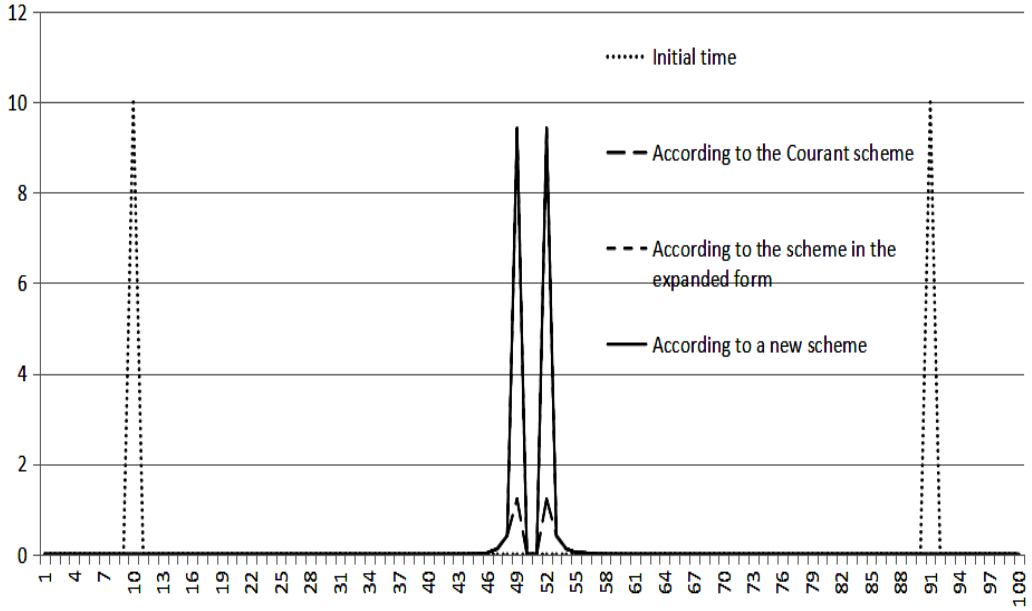

Figure 3.
Solution to the problem of additional mass transfer for schemes (5), (6), and (7) The velocity of motion is zero at two central points. Calculation results for the 41st time step (Courant's parameter is equal to one).

Analyzing the results shown in Fig. 4, we note the appearance of partial conservatism for the Courant scheme. However, conservatism always manifests itself when approaching the zero zone only from one side. It is impossible to organize the symmetry of the solution with respect to the zone of zero velocities when using conditions "⩽" and "⩾." Since calculations according to the options of the Courant scheme (5) occur sequentially, there is always a first and second check when choosing one or another calculation option from two possible ones. The second branch of the calculation will always be involved in the calculation. Both calculation options will never be involved at the same time. That is, the use of non-strict conditions for checking the sign of the velocities will always give rise to asymmetry of solutions. This calculation possibility is absolutely unacceptable and is no longer considered.

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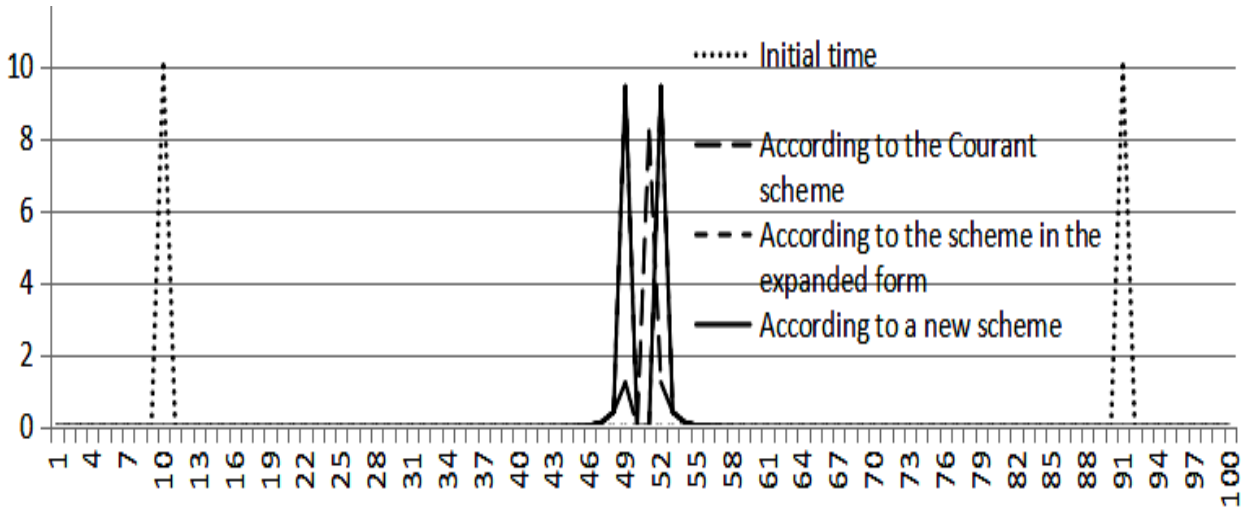

Figure 4.
Solution to the problem of additional mass transfer for schemes (5), (6), and (7). The velocity of motion is zero at two central points. Calculation results for the 41st time step (Courant's parameter is equal to one).

In the next test problem, we structure our calculations so that when the additional mass reaches the central point, the velocity signs switch to their opposites. The results are unusual (Figure 5). First, Courant's scheme becomes more unstable. When the Courant number is one, the instability emerges with an incorrect negative value of the substance concentration. Scheme (6) remains non-conservative. Moreover, the substance appears stuck to the central zone and doesn't move away. The central zone's sign change generates substance infinitely. Only the new scheme (7) proposed for use produces acceptable results. Note that when applying the Courant scheme to this test problem, the jaggedness in the solution disappears only when the Courant number is half or less than the theoretical maximum. At Courant numbers slightly above 0.5, the jaggedness appears smaller and is eventually "overwritten" by the scheme's viscosity. By examining the results of the Courant scheme (5), we can always find the instability of the jaggedness in the first-time steps after the simultaneous velocity sign change at the 42nd time step.

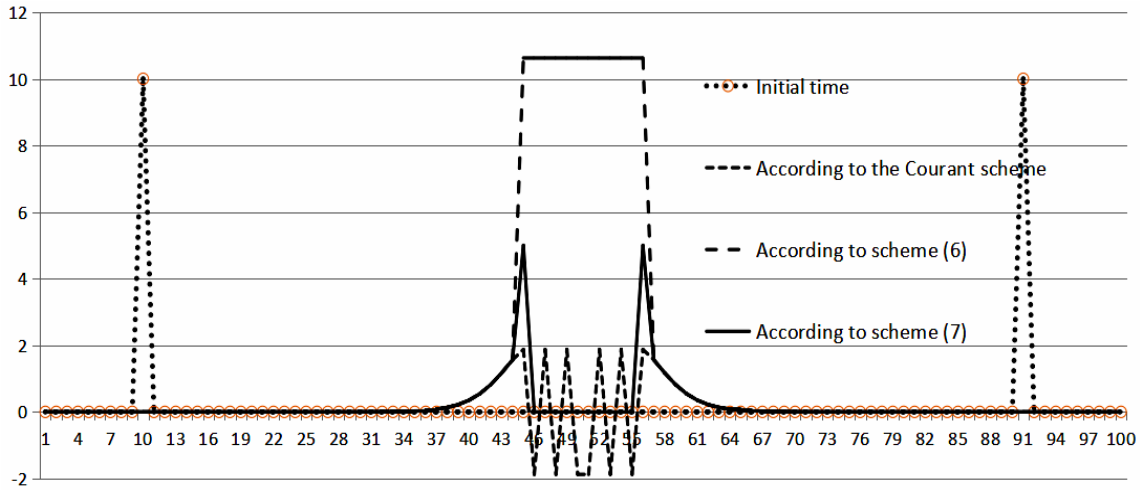

Figure 5.
Results of solving the equation of additional mass transfer using schemes (5), (6), (7).

Edelweiss Applied Science and Technology
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Let us consider how the studied schemes behave when the Courant number is 0.8 (Fig. 6). We picked this Courant number because it's high enough. This way, the scheme's viscosity doesn't show too much. Still, it smooths out the "jaggedness" that appears when we use this scheme (5).

The new scheme (7) proposed gives a conservative result. The Courant scheme (5) and scheme (6) lost their conservatism when transferring additional mass.

Let's redo the calculation, assuming there's a zone of zero velocities at the center (Fig. 7). Scheme (6) and the new scheme (7) proposed for use give a conservative result. Courant's scheme (5) lost its conservatism when transferring additional mass.

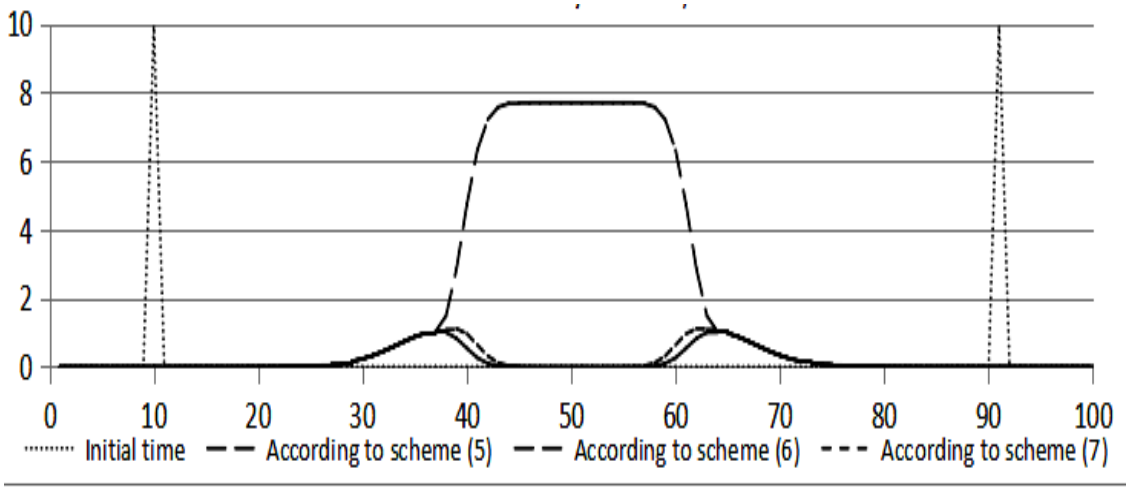

Figure 6.
Results of solving the equation of additional mass transfer using schemes (5), (6), and (7). The Courant number is equal to 0.88 on the 61st step; the change of sign velocities occurs at the 49th step, a zone of zero velocity is absent.

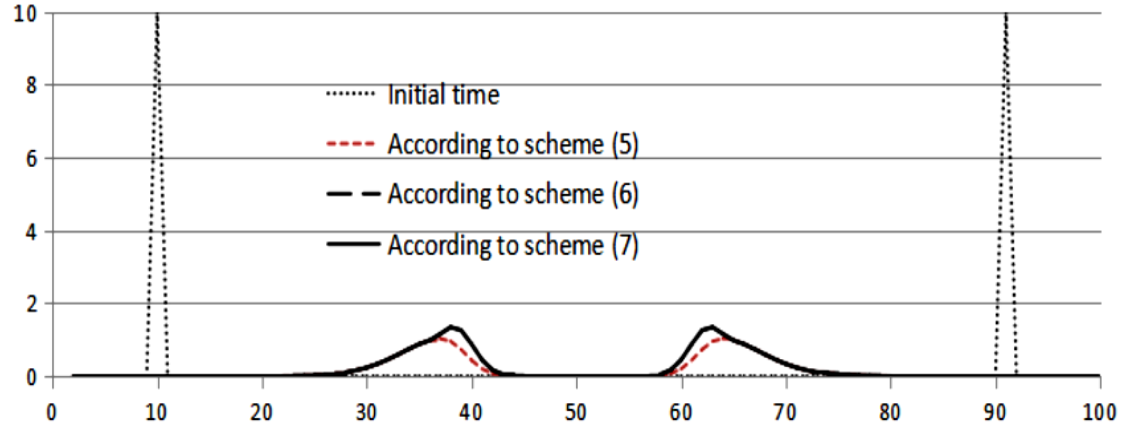

Figure 7.
Results of solving the equation of additional mass transfer using schemes (5), (6), and (7). The Courant number is equal to 0.88 on the 61st step; the change of sign velocities occurs at the 49th step, and there is a zone of zero velocity in the center.

Let us now look at how schemes (5), (6), and (7) can work when moving from a one-dimensional problem to a two-dimensional one. Note that the transition to three-dimensional schemes can also be easily performed.

Let us define a two-dimensional velocity field in the form of two symmetrical circular motions. The velocity modulus in each circular motion is equal to one. The steps in space are also equal to one. Figure 8 shows a field of two contacting circular motions. At the initial time point, in the nodes highlighted

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Vol. 9, No. 7: 162-175, 2025
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with a darker background, the concentration of the substance is set to one. An acceptable solution is expected in the form of circular and symmetrical motions of the additional mass along the corresponding circular contours.

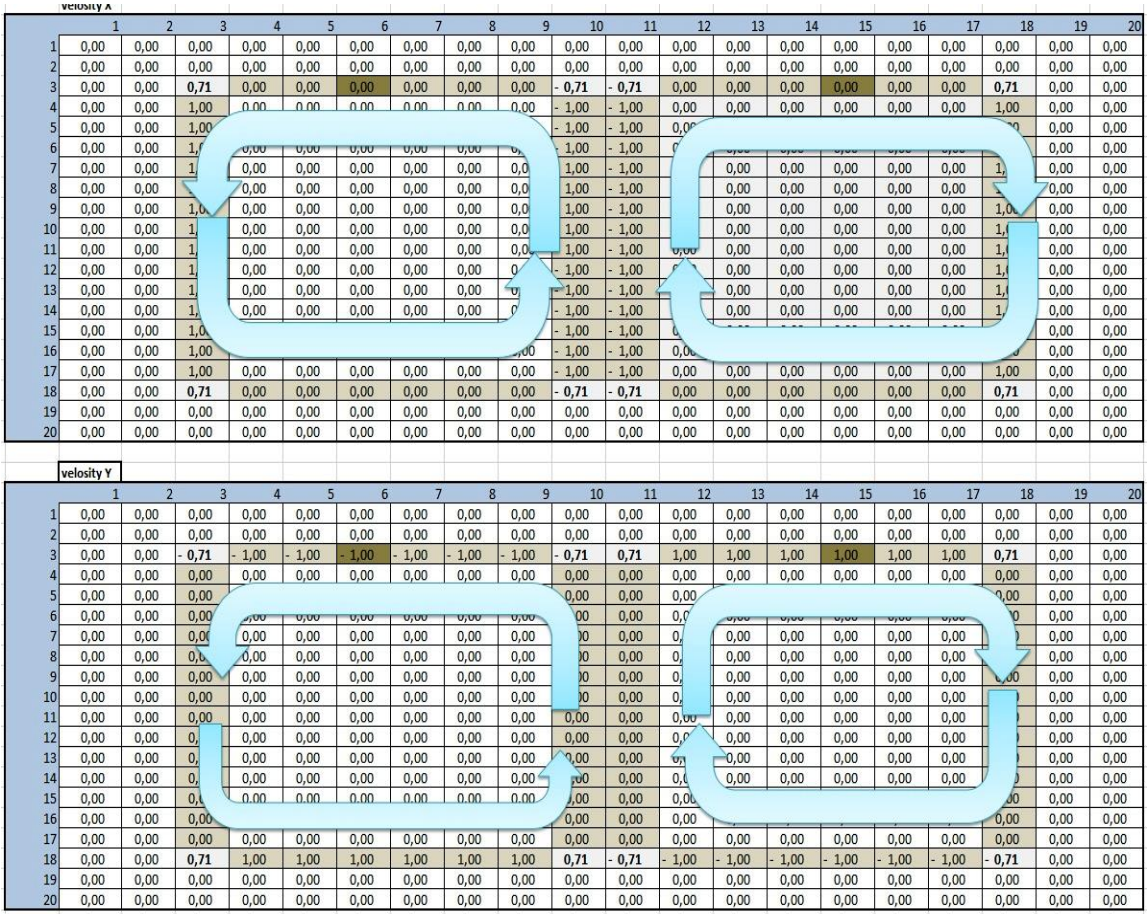

Figure 8.
Velocity field for the test problem of transporting additional mass in two closed symmetrical and contacting circular motions.

All values of the velocity components equal to zero are assigned a small positive value to ensure the complete utilization of schemes (10) and (11) under their strict conditions for selecting the ">" and "<" calculation branches.

Courant's scheme (5) can be generalized to two- and three-dimensional spaces with relative simplicity:

$$\begin{cases} \frac{S_{i,j}^{t+1}-S_{i,j}^{t}}{\Delta t}+\frac{V_i S_{i,j}^{t}-V_{i-1}S_{i-1,j}^{t}}{\Delta x}+\frac{U_{i,j}S_{i,j}^{t}-U_{i,j-1}S_{i,j-1}^{t}}{\Delta y}=0, V_{i,j}>0, U_{i,j}>0 \\ \frac{S_{i,j}^{t+1}-S_{i,j}^{t}}{\Delta t}+\frac{V_{i+1}S_{i+1,j}^{t}-V_i S_{i,j}^{t}}{\Delta x}+\frac{U_{i,j}S_{i,j}^{t}-U_{i,j-1}S_{i,j-1}^{t}}{\Delta y}=0, V_{i,j}<0, U_{i,j}>0 \\ \frac{S_{i,j}^{t+1}-S_{i,j}^{t}}{\Delta t}+\frac{V_i S_{i,j}^{t}-V_{i-1}S_{i-1,j}^{t}}{\Delta x}+\frac{U_{i,j+1}U_{i,j+1}^{t}-U_{i,j}S_{i,j}^{t}}{\Delta y}=0, V_{i,j}>0, U_{i,j}<0 \\ \frac{S_{i,j}^{t+1}-S_{i,j}^{t}}{\Delta t}+\frac{V_{i+1}S_{i+1,j}^{t}-V_i S_{i,j}^{t}}{\Delta x}+\frac{U_{i,j+1}U_{i,j+1}^{t}-U_{i,j}S_{i,j}^{t}}{\Delta y}=0, V_{i,j}<0, U_{i,j}<0 \end{cases} \quad (10)$$

In formulas (10) and (11), $V_{i,j}$ and $U_{i,j}$ are components of the velocity vector along the coordinate axes:

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$$\begin{cases} \frac{S_{i,j}^{t+1}-S_{i,j}^{t}}{\Delta t}+V_{i,j}\frac{S_{i,j}^{t}-S_{i-1,j}^{t}}{\Delta x}+U_{i,j}\frac{S_{i,j}^{t}-S_{i,j-1}^{t}}{\Delta y}+S_{i,j}^{t}\frac{V_{i+1,j}-V_{i,j}}{\Delta x}+S_{i,j}^{t}\frac{U_{i,j+1}-U_{i,j}}{\Delta y}=0, V_{i,j}>0, U_{i,j}>0 \\ \frac{S_{i,j}^{t+1}-S_{i,j}^{t}}{\Delta t}+V_{i,j}\frac{S_{i+1,j}^{t}-S_{i,j}^{t}}{\Delta x}+U_{i,j}\frac{S_{i,j}^{t}-S_{i,j-1}^{t}}{\Delta y}+S_{i,j}^{t}\frac{V_{i,j}-V_{i-1,j}}{\Delta x}+S_{i,j}^{t}\frac{U_{i,j+1}-U_{i,j}}{\Delta y}=0, V_{i,j}<0, U_{i,j}>0 \\ \frac{S_{i,j}^{t+1}-S_{i,j}^{t}}{\Delta t}+V_{i,j}\frac{S_{i,j}^{t}-S_{i-1,j}^{t}}{\Delta x}+U_{i,j}\frac{S_{i,j+1}^{t}-S_{i,j}^{t}}{\Delta y}+S_{i,j}^{t}\frac{V_{i+1,j}-V_{i,j}}{\Delta x}+S_{i,j}^{t}\frac{U_{i,j}-U_{i,j-1}}{\Delta y}=0, V_{i,j}>0, U_{i,j}<0 \\ \frac{S_{i,j}^{t+1}-S_{i,j}^{t}}{\Delta t}+V_{i,j}\frac{S_{i+1,j}^{t}-S_{i,j}^{t}}{\Delta x}+U_{i,j}\frac{S_{i,j+1}^{t}-S_{i,j}^{t}}{\Delta y}+S_{i,j}^{t}\frac{V_{i,j}-V_{i-1,j}}{\Delta x}+S_{i,j}^{t}\frac{U_{i,j}-U_{i,j-1}}{\Delta y}=0, V_{i,j}<0, U_{i,j}<0 \end{cases} \quad (11)$$

(12) presents the new scheme proposed for use:

$$\begin{aligned} &Ux_{t,i,j} = Vx_{t,i,j} \quad if \ Vx_{t,i,j} \geqslant 0, Ux_{t,i,j} = 0 \quad if\, Vx_{t,i,j} \leqslant 0, \\ &Wx_{t,i,j} = Vx_{t,i,j} \quad if\, Vx_{t,i,j} \leqslant 0, Wx_{t,i,j} = 0 \quad if\, Vx_{t,i,j} \geqslant 0, \\ &Uy_{t,i,j} = Vy_{t,i,j} \quad if\, Vy_{t,i,j} \geqslant 0, Uy_{t,i,j} = 0 \quad if\, Vy_{t,i,j} \leqslant 0, \\ &Wy_{t,i,j} = Vy_{t,i} \quad if\, Vy_{t,i,j} \leqslant 0, Wy_{t,i,j} = 0 \quad if\, Vy_{t,i,j} \geqslant 0, \\ &S_{t+1,i,j} = S_{t,i,j} - \frac{\Delta t}{\Delta x}\begin{bmatrix} Ux_{t,i,j}\left(S_{t,i,j}-S_{t,i-1,j}\right)+S_{t,i,j}\left(Ux_{t,i+1,j}-Ux_{t,i,j}\right) \\ + \\ Wx_{t,i,j}\left(S_{t,i+1,j}-S_{t,i,j}\right)+S_{t,i,j}\left(Wx_{t,i,j}-Wx_{t,i-1,j}\right) \end{bmatrix} \\ &-\frac{\Delta t}{\Delta y}\begin{bmatrix} Uy_{t,i,j}\left(S_{t,i,j}-S_{t,i,j-1}\right)+S_{t,i,j}\left(Uy_{t,i,j+1}-Uy_{t,i,j}\right) \\ + \\ Wy_{t,i,j}\left(S_{t,i,j+1}-S_{t,i,j}\right)+S_{t,i,j}\left(Wy_{t,i,j}-Wy_{t,i,j-1}\right) \end{bmatrix} \end{aligned} \quad (12)$$

where $Ux_{t,i,j}$, $Wx_{t,i,j}$, $Uy_{t,i,j}$, $Wy_{t,i,j}$ are new variables, calculated through the value of the substance's motion velocity depending on the sign of this velocity.

The Courant scheme produces uneven and insufficient results. It can show negative values, and some of the substance tends to stick at the corners during circular motion (Fig. 9). The Courant parameter is set to 0.99. It is interesting that the Courant scheme does not withstand the greatest theoretical value of the Courant parameter. The instability of the "jaggedness" disappears only when the Courant parameter is 0.5. Yet, other problems remain even at a lower Courant number.

The application of scheme (11) yields a superior result in comparison to that obtained by scheme (10). Nevertheless, the primary disadvantage of scheme (11) is its incompatibility with practical calculations. It lacks conservatism. So, because the substance keeps increasing, the calculation will lead to an emergency stop. Substances in the solution area reached 5.20 (as shown in Fig. 10)!

The expanded structure of scheme (11) made it possible to construct scheme (12) free of all errors and shortcomings (Fig. 11).

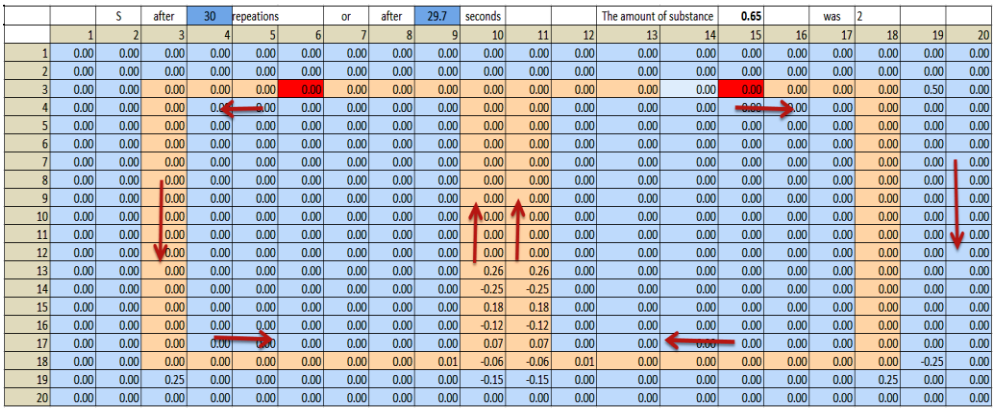
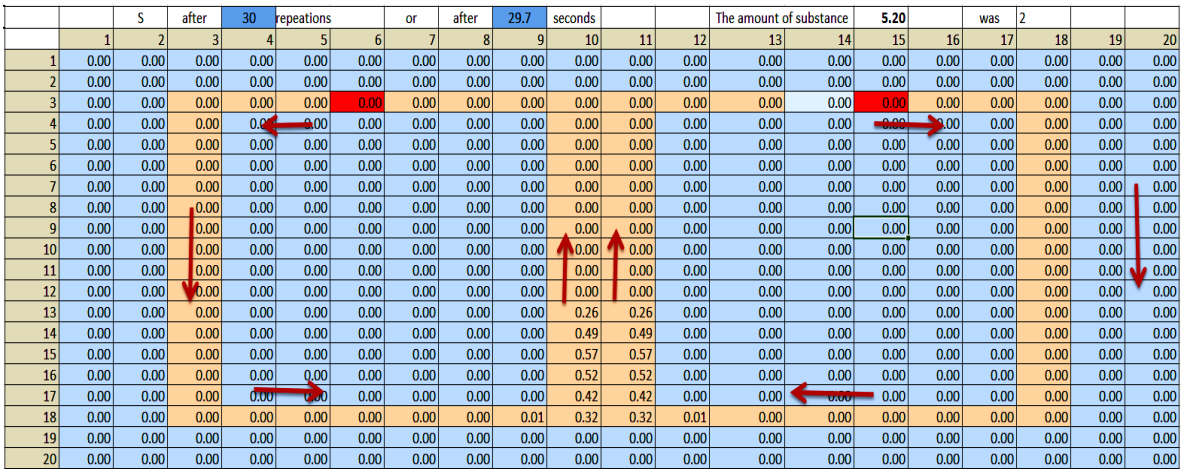


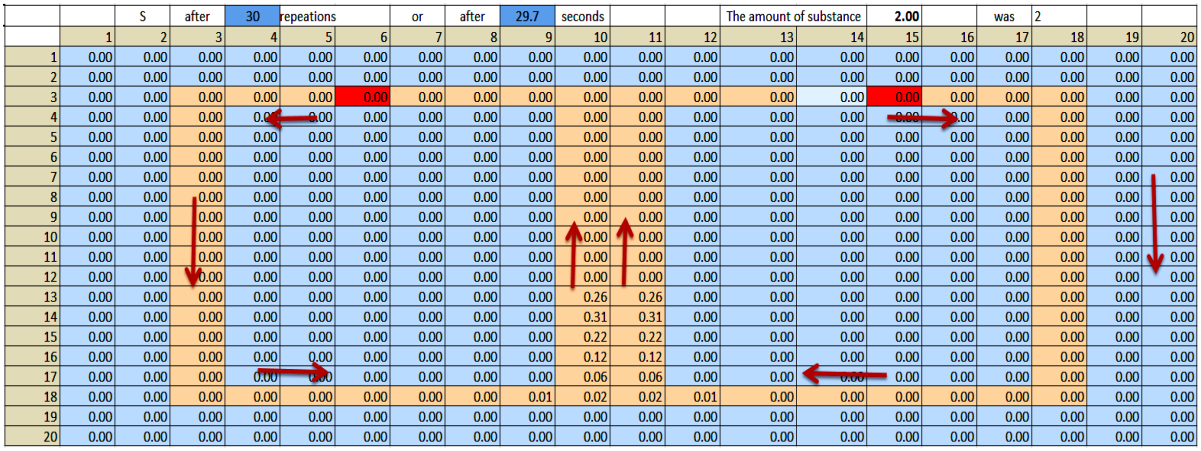
Figure 9.
Result of solving the substance transfer equation (1) using the Courant scheme.

Edelweiss Applied Science and Technology
ISSN: 2576-8484
Vol. 9, No. 7: 162-175, 2025
DOI: 10.55214/25768484.v9i7.8557



		S	after	30	repeations		or	after	29.7	seconds			The amount of substance		5.20		was	2		
	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
1	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
2	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
3	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
4	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
5	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
6	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
7	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
8	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
9	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
10	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
11	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
12	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
13	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.26	0.26	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
14	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.49	0.49	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
15	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.57	0.57	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
16	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.52	0.52	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
17	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.42	0.42	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
18	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.01	0.32	0.32	0.01	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
19	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
20	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00

Figure 10.
Result of solving the substance transfer equation (2) according to scheme (11).



		S	after	30	repeations		or	after	29.7	seconds			The amount of substance		2.00		was	2		
	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
1	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
2	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
3	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
4	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
5	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
6	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
7	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
8	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
9	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
10	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
11	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
12	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
13	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.26	0.26	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
14	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.31	0.31	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
15	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.22	0.22	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
16	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.12	0.12	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
17	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.06	0.06	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
18	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.01	0.02	0.02	0.01	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
19	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
20	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00

Figure 11.
Result of solving the substance transfer equation (2) according to scheme (12).

Let us compare the calculation results obtained using schemes (10), (11), and (12) along the supposed trajectory of substance. The results obtained from schemes (10) and (11) are not acceptable.

Edelweiss Applied Science and Technology
ISSN: 2576-8484
Vol. 9, No. 7: 162-175, 2025
DOI: 10.55214/25768484.v9i7.8557

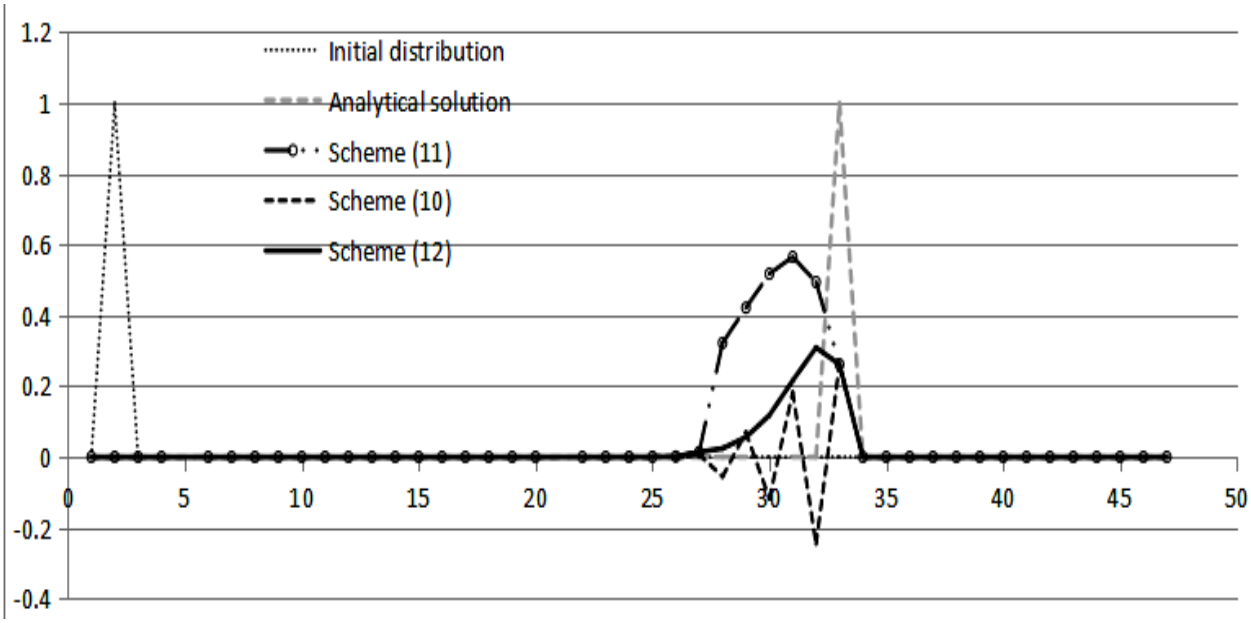


Figure 12.
Results of comparisons of solutions to the substance transfer equations (1) - (2) according to schemes (10), (11), and (12).

The results of scheme (12) show scheme viscosity. It is clear that the substance transfer velocities at turns are less than one. When both velocity components at the corner points equal one, the analytical solution matches the approximate solution from scheme (12). Then the velocity modulus at the corner points would be equal to $\sqrt{2}$, and not to 1. We have not yet solved this little paradox. However, it does not have a significant impact on the consumer capabilities of scheme (12).

3. Conclusions

The proposed finite-difference schemes (7, 12) overcome the calculation issues inherent in the common Courant scheme. Numerical experiments show that other finite-difference schemes suffer from the same problems as the Courant scheme. Notably, the proposed scheme (12) remains stable in the two-dimensional case when the Courant parameter equals one, which is not true for other known and studied schemes.

Generalizing Scheme (12) to three-dimensional space is relatively easy. Implementing the proposed scheme requires minimal changes to the codes of fluid mechanics models. In plane and spatial problems, points where the sign of transverse velocities changes often occur when the cross-section of the flow changes. This leads to non-conservative calculations. Salokhiddinov, et al. [31] discusses the loss of conservatism in calculations and presents ways to reduce it. Refining the mesh also reduces non-conservativeness [32]. The lower the absolute velocities around the point of sign change, the smaller the loss of conservatism [33]. However, "less" does not mean "zero." Schemes (7, 12) ensure exact conservatism. The practical result is that the loss of conservatism in calculations is eliminated, allowing for larger time steps in most problems.

Transparency:

The authors confirm that the manuscript is an honest, accurate, and transparent account of the study; that no vital features of the study have been omitted; and that any discrepancies from the study as planned have been explained. This study followed all ethical practices during writing.

Edelweiss Applied Science and Technology
ISSN: 2576-8484
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DOI: 10.55214/25768484.v9i7.8557

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Edelweiss Applied Science and Technology
ISSN: 2576-8484
Vol. 9, No. 7: 162-175, 2025
DOI: 10.55214/25768484.v9i7.8557