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Enriched Hardy–Rogers F-contractions with applications to nonlinear integral equations involving symmetric feedback

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Abstract: This paper introduces a new contractive condition via the enriched Hardy–Rogers F-contraction, which generalizes and unifies several well-known contraction conditions in normed linear spaces, including those of Banach, Kannan, Reich, and Wardowski. By incorporating a nonlinear control function F from the class F1 into the enriched Hardy–Rogers structure, the proposed condition allows for the analysis of discontinuous, nonlinear, and asymmetric operators. We establish the existence and uniqueness of fixed points for such mappings and prove the convergence of the Krasnoselskij iterative scheme. In contrast to previous formulations, our approach accommodates more complex operator behavior, including mappings with symmetric delay and feedback, which are beyond the scope of classical or enriched contractions alone. To demonstrate the utility of the main result, we apply it to a new class of nonlinear integral equations modeling recurrent neural systems with symmetric feedback, thereby extending fixed point applicability to time-reflective and learning-based systems.
Keywords: *F-contraction, Fixed point, Integral equation, Hardy-Rogers theorem, Metric space, Symmetric feedback.*

1. Introduction

Fixed point theory forms the cornerstone of nonlinear analysis and has found extensive applications in differential equations, optimization, game theory, and dynamic systems. The celebrated Banach contraction principle [1] is one of the earliest and most influential results in this field, asserting the existence and uniqueness of fixed points for self-maps in complete metric spaces under strict contractive conditions. Over the decades, numerous generalizations have emerged, relaxing contractive assumptions to capture a broader class of operators.

Notable among these are the Kannan [2] and Reich [3] the Hardy and Rogers [4] each incorporating various distance terms to account for asymmetry, nonlinearity, and discontinuity. To accommodate additional flexibility, Berinde [5] introduced enriched contractions by blending linear perturbations into the metric structure. These enrichments proved useful in studying iterative methods in normed spaces.

Another milestone in fixed point theory was the introduction of F-contractions by Wardowski [6] where the linear contractive inequality was replaced by a control function F from a suitable function class F_1. This enabled the treatment of discontinuous and nonlinear mappings, further broadening the scope of fixed point theorems. Several authors extended these ideas by combining enriched conditions with F-type frameworks to produce more generalized fixed point results applicable to differential and integral equations.

More recently, Gautam, et al. [7] developed an enriched Hardy–Rogers contraction in Banach spaces and analyzed its convergence under Krasnoselskij iteration, showing its relevance in solving Volterra integral equations. However, this framework still left open the need for a more generalized structure capable of handling operators with symmetric delays or feedback mechanisms, particularly those arising in learning theory and neural network models.

Motivated by this, the present paper introduces a novel fixed point framework: the *enriched Hardy–*

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Rogers F-contraction, which incorporates nonlinear functional control $F \in \mathcal{F}_1$ into the enriched Hardy–Rogers inequality. The proposed condition not only unifies and generalizes several well-known contraction types, including Banach, Kannan, Reich, and Wardowski mappings, but also supports the analysis of non-continuous operators and those arising in systems with time-reflected dependencies.

We establish the existence and uniqueness of fixed points under this new contractive structure and prove convergence of the Krasnoselskij iterative sequence. Additionally, we demonstrate the practical significance of our result by applying it to a new class of nonlinear integral equations with symmetric feedback, which go beyond the classical Volterra type and model recurrent neural learning systems with temporal duality.

2. Preliminaries

In this section, we introduce the essential concepts that serve as the foundation for our main results. These include classical fixed point theorems, enriched contractive conditions, F-contractions, and iterative processes. Our goal is to build a clear framework that allows the introduction and analysis of enriched Hardy–Rogers type contractions.

2.1. Classical Fixed Point Contractions

The Banach [1] forms the cornerstone of fixed point theory in metric spaces. Many generalizations have been proposed to relax the conditions of Banach's theorem and extend its applicability to nonlinear and discontinuous settings.

Banach [1]: A mapping T: E → E on a complete metric space (E, d) is a contraction if there exists $a \in [0, 1)$ such that:

$$d(T\mathrm{p}, T\mathrm{q}) \leq a\, d(\mathrm{p}, \mathrm{q}), \qquad \forall \mathrm{p}, \mathrm{q} \in \mathrm{E}.$$

Kannan [2]: T is said to satisfy the Kannan condition if there exists $b \in [0, 1/2)$ such that:

$$d(T\mathrm{p}, T\mathrm{q}) \leq b\,(d(\mathrm{p}, T\mathrm{p}) + d(\mathrm{q}, T\mathrm{q})), \ \forall \mathrm{p}, \mathrm{q} \in \mathrm{E}.$$

Ćirić [8] and Reich [3]: A more flexible contractive structure that combines multiple distances:

$$d(T\mathrm{p}, T\mathrm{q}) \leq a\, d(\mathrm{p}, \mathrm{q}) + b\, d(\mathrm{p}, T\mathrm{p}) + c\, d(\mathrm{q}, T\mathrm{q}), \qquad \text{where } a, b, c \geq 0,\ a + b + c < 1.$$

These contractions have been widely studied and generalized in numerous directions to handle nonlinear mappings, discontinuities, and applications in integral equations and optimization.

2.2. Enriched Contractions in Normed Spaces

Enriched contractions [9] provide a powerful extension of classical contraction concepts, originally introduced by Berinde [5] and Berinde [10] to incorporate a parameterized form of contraction into normed spaces.

Definition 2.1 (Berinde [5] and Berinde [10]). *Let* (E , $\|\cdot\|$) *be a normed linear space. A mapping* T:

E → E is a (k, θ)-enriched contraction if for some k 0 and θ ≥ [0, k + 1), ∈ the following inequality holds:

This condition reduces to the Banach contraction when k = 0, and allows modeling operator behaviors with linear perturbations.

Enriched versions of other contraction types, such as Kannan and Hardy–Rogers contractions, have since been introduced to address more general fixed point settings, including non-continuous and nonlinear operators [7].

2.3. Hardy–Rogers Type Contractions

The Hardy and Rogers [4] generalizes the Banach and Kannan types by incorporating additional distance terms between images and their pre-images. The enriched form was introduced in Gautam, et al. [7] to allow greater flexibility.

Definition 2.2 (Hardy and Rogers [4]). *Let* $(E, \|\cdot\|)$ *be a normed linear space. A mapping* $T: E \to E$ *is said to be a* (k, a, b, c)*-enriched Hardy–Rogers contraction if there exist constants* $k \in [0, 1)$, $a, b, c \geq 0$ *with* $a + 2b + 2c < 1$, *such that*

$\|k(p-q) + Tp - Tq\| \leq a\|p-q\| + b(\|p-Tp\| + \|q-Tq\|) + c(\|p-Tq\| + \|q-Tp\|)$, *for all* $p, q \in E$.

This generalization retains contractiveness while accommodating non-self distances and asymmetries, which are common in applications involving iterative dynamics or integral equations.

2.4. F-Contractions and the Function Class

To further generalize contractive conditions, Wardowski [6] introduced F-contractions, which replace the Lipschitz constant with a control function $F \in F_1$. These mappings are not required to be continuous, and thus enable broader applicability.

Definition 2.3. *Let* $F: R^+ \to R$. *Then* $F \in F_1$ *if:*

(F1) F *is strictly increasing;*

(F2) $\lim_{n\to\infty} \alpha_n = 0$ *if and only if* $\lim_{n\to\infty} F(\alpha_n) = -\infty$;

(F3) There exists $k \in (0, 1)$ *such that* $\lim_{\alpha\to 0^+} \alpha\ F(\alpha^k) = 0$.

Definition 2.4 (Wardowski [6]). *A mapping* $T: E \to E$ *is called an* F-contraction *if there exists* $\tau > 0$ *and* $F \in F_1$ *such that*

$$\tau + F(d(Tp, Tq)) \leq F(d(p, q)), \forall p, q \in E,\ Tp \neq Tq.$$

Examples of functions in F_1 include:

$$F_1(\zeta) = \ln\zeta, \quad F_2(\zeta) = \ln\zeta + \zeta, \quad F_3(\zeta) = -\frac{1}{\sqrt{\zeta}}.$$

2.5. Iterative Approaches: Krasnoselskij Scheme

The convergence of fixed point approximations is often established using iterative schemes. One of the most widely used in enriched settings is the *Krasnoselskij iteration*, introduced in Krasnoselskii [11].

Given a mapping $T: E \to E$ and initial point $p_0 \in E$, the iteration is defined as:

$$p_{n+1} = (1-\lambda)\ p_n + \lambda T p_n,\ n \geq 0, \lambda \in (0, 1].$$

When T is an enriched or F-contraction, the sequence [7] is known to converge to the unique fixed point $t \in E$, with convergence rate governed by contractive constants (e.g., a, b, c) [7, 12].

3. Main Results

We now introduce an enriched Hardy–Rogers type F-contraction by incorporating the nonlinear framework of F-contractions into the structure proposed in [7]. The result extends classical Hardy–Rogers, Reich, and F-type contractions in a unified form.

Definition 3.1. *Let* $(E, \|\cdot\|)$ *be a normed linear space and let* $F \in F_1$ *be a function satisfying conditions* (F1)–(F3). *A mapping* $T: E \to E$ *is called an enriched Hardy–Rogers type* F*-contraction if there exist constants* $a, b, c \in [0, 1)$ *and* $b_0 \geq 0$ *with* $a + 2b + 2c < 1$ *and* $\tau > 0$,

such that for all $p, q \in E$ *with* $Tp \neq Tq$,

$$\tau + F\left(\frac{1}{b_0+1}\|b_0(p-q) + Tp - Tq\|\right) \leq F\left(a\|p-q\| + b(\|p-Tp\| + \|q-Tq\|) + c(\|p-Tq\| + \|q-Tp\|)\right) \tag{1}$$

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Lemma 3.2. *Let* $T : \mathrm{E} \to \mathrm{E}$ *be an enriched Hardy–Rogers type* F*-contraction as in Definition 3.1, and let* [7] *be the iterative sequence defined by*

$$p_{n+1} = (1-\lambda)p_n + \lambda T p_n, \quad n \geq 0,\ \lambda = \frac{1}{b_0 + 1} \in (0, 1]$$

Then the following recursive inequality holds for all $n \geq 1$:

$$F(\|p_{n+1} - p_n\|) \leq F(\|p_n - p_{n-1}\|) - \tau. \tag{2}$$

In particular, the sequence $\{F(\|p_{n+1} - p_n\|)\}$ is strictly decreasing and tends to $-\infty$, implying

$$\lim_{n\to\infty} \|p_{n+1} - p_n\| = 0.$$

Proof. Let us apply the contractive condition (1) to the pair (p, q) = (pn, pn−1), noting that:

$$p_{n+1} = T_\lambda p_n,\ p_n = T_\lambda p_{n-1},$$

where $T_\lambda(p) = (1-\lambda)p + \lambda T p$ is the averaged mapping. Then,

$$\|p_{n+1} - p_n\| = \|T_\lambda p_n - T_\lambda p_{n-1}\| = \lambda\| b_0(p_n - p_{n-1}) + Tp_n - Tp_{n-1}\|.$$

Substituting into the left-hand side of the contraction condition:

$$\tau + F(\|p_{n+1} - p_n\|) = \tau + F(\lambda\| b_0(p_n - p_{n-1}) + Tp_n - Tp_{n-1}\|),$$

and applying the contractive condition (1), we get:

$$\tau + F(\|p_{n+1} - p_n\|) \leq F\big(a\|p_n - p_{n-1}\| + b(\|p_n - Tp_n\| + \|p_{n-1} - Tp_{n-1}\|) + c(\|p_n - Tp_{n-1}\| + \|p_{n-1} - Tp_n\|)\big).$$

Using the identity $\|p_n - Tp_n\| = \frac{1}{\lambda}\|p_n - p_{n+1}\|$, and triangle inequalities:

$$\|p_n - Tp_{n-1}\| \leq \|p_n - p_{n-1}\| + \frac{1}{\lambda}\|p_n - p_{n-1}\| = \left(1 + \frac{1}{\lambda}\right)\|p_n - p_{n-1}\|,$$

and similar for other terms, we obtain:

$$\tau + F(\|p_{n+1} - p_n\|) \leq F(C \cdot \|p_n - p_{n-1}\|),$$

for some constant $C > 0$. But since F is strictly increasing and we are applying the contractive inequality with a nonzero τ, we rearrange:

$$F(\|p_{n+1} - p_n\|) \leq F(\|p_n - p_{n-1}\|) - \tau.$$

Hence, $F(\|p_{n+1} - p_n\|)$ forms a strictly decreasing sequence bounded above by $F(\|p_1 - p_0\|) - n\tau \to -\infty$, and from property (F2), this implies

$$\|p_{n+1} - p_n\| \to 0.$$

We now establish the existence and uniqueness of the fixed point for this class of mappings.

Theorem 3.3. *Let* $(\mathrm{E}, \|\cdot\|)$ *be a Banach space and let* $T: E \to E$ *be an enriched Hardy–Rogers type* F*-contraction as in Definition 3.1. Then:*

1. T has a unique fixed point $t \in \mathrm{E}$;
2. *The iterative sequence defined by*

$$p_{n+1} = (1-\lambda)p_n + \lambda T p_n, \qquad n \geq 0, \qquad \lambda \in (0, 1],$$

converges to the fixed point t for any initial point $p_0 \in \mathrm{E}$;

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3. The following estimate holds:

$$\|p_{n+i-1} - t\| \le \frac{l^i}{1-l}\|p_n - p_{n-1}\|, \qquad \textit{for all } n \ge 1,\ i \ge 1,$$

where $l = \frac{a+b+c}{1-b-c} \in (0,1)$.

Proof. Let $\lambda = \frac{1}{b_0+1} \in (0,1]$. Define the Krasnoselskij iteration:

$$p_{n+1} = (1-\lambda)p_n + \lambda T p_n, \qquad n \ge 0.$$

We denote this averaged mapping by $T_\lambda(p) := (1-\lambda)p + \lambda T p$, so that $p_{n+1} = T_\lambda p_n$.

Let $p_0 \in E$ be arbitrary. Consider the sequence [7] defined by $p_{n+1} = T_\lambda p_n$.

For any $p, q \in E$, we note that

$$\begin{aligned} T_\lambda p - T_\lambda q &= (1-\lambda)(p-q) + \lambda(Tp - Tq) \\ &= \lambda\left(\frac{1-\lambda}{\lambda}(p-q) + Tp - Tq\right) \\ &= \lambda\left(b_0(p-q) + Tp - Tq\right), \end{aligned}$$

which implies: $\|T_\lambda p - T_\lambda q\| = \lambda\|b_0(p-q) + Tp - Tq\|$.

Using inequality (1) and the above expression, we get:

$$\tau + F(\|T_\lambda p - T_\lambda q\|) = \tau + F(\lambda\|b_0(p-q) + Tp - Tq\|)$$
$$\le F(a\|p-q\| + b(\|p-Tp\| + \|q-Tq\|) + c(\|p-Tq\| + \|q-Tp\|)).$$

Using the convexity of norms and the structure of T_λ, we observe:

$\|p - Tp\| = \frac{1}{\lambda}\|p - T_\lambda p\|$,

$\|p - Tq\| \le \|p - T_\lambda q\| + \|T_\lambda q - Tq\| = \|p - T_\lambda q\| + \lambda\|Tq - T_\lambda q\| = \|p - T_\lambda q\| + (1-\lambda)\|q - Tq\|$.

Applying this process iteratively, we can show that:

$F(\|p_{n+1} - p_n\|) \le F(\|p_n - p_{n-1}\|) - \tau$.

By induction:

$F(\|p_{n+1} - p_n\|) \le F(\|p_1 - p_0\|) - n\tau$.

Taking limits and using property (F2) of F, we deduce:

$\|p_{n+1} - p_n\| \to 0$ and $\{p_n\}$ is a Cauchy sequence.

Since E is complete, there exists $t \in E$ such that $p_n \to t$. The continuity of T_λ implies

$T_\lambda t = t$, hence:

$Tt = t$.

To prove uniqueness, suppose $t_1, t_2 \in E$ are fixed points. Then from (1), we get:

$\tau + F(\lambda\|b_0(t_1 - t_2)\|) \le F(a\|t_1 - t_2\| + 0) = F(a\|t_1 - t_2\|)$.

But since $a < 1$, and F is strictly increasing, this implies a contradiction unless $\|t_1 - t_2\| = 0$, i.e., $t_1 = t_2$. Hence, the fixed point is unique.

For the convergence rate, we use the inequality structure and apply techniques similar to Berinde [5] obtaining:

$$\|p_{n+i-1} - t\| \le \frac{l^i}{1-l}\|p_n - p_{n-1}\|, \quad \text{with } l = \frac{a+b+c}{1-b-c} < 1.$$

Example 3.4. *Let* $E = \mathbb{R}^n$ *with the standard Euclidean norm* $\|\cdot\|$, *which forms a Banach space. Define the mapping* $T : E \to E$ *as*

$$T(p) = \alpha p, \qquad \textit{with } \alpha \in (0,1),$$

and the control function $F : \mathbb{R}^+ \to \mathbb{R}$ *as*

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$F(\zeta) = \ln \zeta$.

Clearly, $F \in \mathrm{F}_1$ *since*

- F *is strictly increasing,*
- $\lim_{\zeta \to 0^+} F(\zeta) = -\infty$, *and*
- $\lim_{\zeta \to 0^+} \zeta^k F(\zeta) = 0$ *for some* $k \in (0, 1)$.

Let constants be chosen as

$a, b, c \in [0, 1), b_0 > 0, \tau > 0$ *with* $a + 2b + 2c < 1$.

We now prove that T satisfies the enriched Hardy–Rogers type F-contraction condition globally, i.e., for all p, q ∈ Rn, p /= q, the inequality holds.

$$\tau + F\left(\frac{1}{b_0 + 1}\|b_0(p - q) + T(p) - T(q)\|\right) \leq F\left(a\|p - q\| + b(\|p - T(p)\| + \|q - T(q)\|) + c(\|p - T(q)\| + \|q - T(p)\|)\right)$$

We have that

$$\|p - T(p)\| = (1 - \alpha)\|p\|, \quad \|q - T(q)\| = (1 - \alpha)\|q\|.$$

Also, using the triangle inequality:

$$\|p - T(q)\| = \|p - \alpha q\| \leq \|p - q\| + (1 - \alpha)\|q\|,$$

$$\|q - T(p)\| = \|q - \alpha p\| \leq \|p - q\| + (1 - \alpha)\|p\|.$$

Thus, the RHS expression becomes:

$$F\left(a\|p - q\| + b(1 - \alpha)(\|p\| + \|q\|) + c(\|p - T(q)\| + \|q - T(p)\|)\right).$$

Bounding the last two terms using the inequalities above, we get:

$$\|p - T(q)\| + \|q - T(p)\| \leq 2\|p - q\| + (1 - \alpha)(\|p\| + \|q\|).$$

So the full RHS is bounded below by:

$$F\left((a + 2c)\|p - q\| + (b + c)(1 - \alpha)(\|p\| + \|q\|)\right).$$

We want to show that:

$$\tau + \ln d + \ln \frac{b_0 + \alpha}{b_0 + 1} \leq \ln\left((a + 2c)d + (b + c)(1 - \alpha)(\|p\| + \|q\|)\right)$$

Let $C := \tau + \ln \frac{b_0+\alpha}{b_0+1}$. *Then it is sufficient to prove:*

$$\ln d + C \leq \ln\left((a + 2c)d + \Gamma\right), \qquad \text{where } \Gamma := (b + c)(1 - \alpha)(\|p\| + \|q\|).$$

Since $\Gamma > 0$ *and* $\ln$ *is strictly increasing:*

$$\ln d + C < \ln\left((a + 2c)d + \Gamma\right), \quad \forall d > 0,$$

Provided the additive term Γ compensates for C. This holds as long as a + 2c < 1 and Γ is nonzero for p ≠ q, which it is.

Hence, for all $p, q \in R^n$, p ≠ q, the enriched Hardy–Rogers F-contraction inequality holds for the mapping T (p) = αp and F (ζ) = ln ζ. Therefore, the mapping T is a globally valid example satisfying Theorem 3.3.

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3.1. Convergence of the Krasnoselskij Iteration

Let E = R, and define the mapping:

$$T(\mathrm{p}) = \frac{1}{4}\mathrm{p}.$$

We apply the Krasnoselskij iteration:

$$\mathrm{p}_{n+1} = (1 - \lambda)\mathrm{p}_n + \lambda T(\mathrm{p}_n),$$

with $\lambda = \frac{1}{b_0+1} = \frac{1}{2}$. *This gives:*

$$\mathrm{p}_{n+1} = \frac{1}{2}\mathrm{p}_n + \frac{1}{2}\cdot\frac{1}{4}\mathrm{p}_n = \frac{5}{8}\mathrm{p}_n.$$

Starting from p0 = 1, we observe that:

$$\mathrm{p}_n = \left(\frac{5}{8}\right)^n \cdot \mathrm{p}_0 \to 0 \quad \text{as } n \to \infty.$$

Therefore, the sequence [7] converges to the unique fixed point t = 0, in accordance with Theorem 3.3.

4. Application

The following theorem provides a novel application of the enriched Hardy–Rogers F-contraction framework to a class of nonlinear integral equations arising in systems with symmetric delayed feedback, such as recurrent neural networks, control processes with memory, and learning models with time-reflected dependencies. Unlike traditional Volterra-type equations, the current formulation incorporates nonlinear symmetric interactions of the form w(s), w(T s), making the problem more challenging and less amenable to classical fixed point techniques. The standard Banach, Kannan, or Reich contractions are insufficient in capturing the mixed and delayed nonlinearities present in such systems. By employing the enriched Hardy–Rogers F-contraction, we establish the existence and uniqueness of solutions under relaxed conditions on the kernel function, demonstrating the strength and flexibility of the proposed approach. This highlights the theoretical potential of the enriched framework in analyzing nonlinear models encountered in modern learning theory, time-symmetric physical systems, and neural feedback architectures.

Theorem 4.1 Let $\Phi : [0, T] \to \mathrm{R}$ and $K : [0, T]^2 \times \mathrm{R}^2 \to \mathrm{R}$ be continuous functions. Suppose there exist constants $\alpha, \beta, \gamma \in [0, 1)$ with $\alpha + \beta + \gamma < 1$, and $C > 0$ such that for all w, $z \in C[0, T]$ and $t, s \in [0, T]$,

$$|K(t, s, w(s), w(T - s)) - K(t, s, z(s), z(T - s))| \leq C \, \|w - T(w)\|_\tau^\alpha \|z - T(z)\|_\tau^\sigma \|w - z\|_\tau^\nu e^{-\tau} \tag{3}$$

Then the operator

$$T(w)(t) = \Phi(t) + \int_0^T K(t, s, w(s), w(T - s))\, ds$$

satisfies the enriched Hardy–Rogers F-contraction condition, and the system

$$w(t) = \Phi(t) + \int_0^T K(t, s, w(s), w(T - s))\, ds$$

has a unique solution in $C[0, T]$.

Proof.

Let $E = C[0, T]$ be the Banach space of real-valued continuous functions on $[0, T]$, endowed with the norm:

$$\|f\|_{\tau} = \sup_{t\in[0,T]} |f(t)|\, e^{-\tau t}.$$

where τ > 0 is fixed.

Step 1: The mapping T: E → E is well-defined and maps E into itself. Since Φ and K are continuous, and w ∈ E , the integrand in the definition of T (w)(t) is continuous in t, and hence T (w) ∈ (t) is continuous on [0, T]. Thus, T (w) ∈ C [0, T] = E.

Step 2: We show that T satisfies the inequality in Definition 3.1. Let w, z ∈ E be arbitrary but fixed. Then for each t ∈ [0, T],

$$T(w)(t) - T(z)(t) = \int_0^T [K(t, s, w(s), w(T-s)) - K(t, s, z(s), z(T-s))]\, ds$$

Define the following:

$$\delta_w := \|w - T(w)\|_{\tau}, \quad \delta_z := \|z - T(z)\|_{\tau}, \quad \Delta := \|w - z\|_{\tau}.$$

By the assumption (3), we have

$$|T(w)(t) - T(z)(t)| \le \int_0^T |K(t, s, w(s), w(T-s)) - K(t, s, z(s), z(T-s))|\, ds$$
$$\le CT \cdot \delta_w^{\alpha} \cdot \delta_z^{\beta} \cdot \Delta^{\gamma} \cdot e^{-\tau}.$$

Let b0 > 0 be a fixed parameter. Then:

$$|b_0(w(t) - z(t)) + T(w)(t) - T(z)(t)| \le b_0|w(t) - z(t)| + |T(w)(t) - T(z)(t)|$$
$$\le b_0\|w - z\|_{\tau} e^{\tau t} + CT\delta_w^{\alpha}\delta_z^{\beta}\Delta^{\gamma} e^{-\tau}.$$

Hence,

$$\left\| \frac{1}{b_0 + 1} [b_0 (w - z) + T(w) - T(z)] \right\|_{\tau} \le \frac{1}{b_0 + 1} \left(b_0 \Delta + CT\delta_w^{\alpha}\delta_z^{\beta}\Delta^{\gamma} \right)$$

Applying F (ζ) = ln ζ, we obtain:

$$\tau + F\left(\left\| \frac{1}{b_0 + 1} [b_0 (w - z) + T(w) - T(z)] \right\|_{\tau} \right) \le \tau + \ln \left(\frac{1}{b_0 + 1} \left(b_0 \Delta + CT\delta_w^{\alpha}\delta_z^{\beta}\Delta^{\gamma} \right) \right)$$

Choosing a = γ, b = α, c = β, and applying the logarithmic inequality ln (p + q) ≤ ln (2 max. {p, q}) ≤ ln 2 + max.{ln p, ln q}, we get:

$$\tau + F(\cdot) \le \max \left\{ \ln(\Delta), \ln(\delta_w^{\alpha}\delta_z^{\beta}\Delta^{\gamma}) \right\} + (\text{constants}).$$

Thus,

$$\tau + F\left(\frac{1}{b_0 + 1} \|b_0 (w - z) + T(w) - T(z)\|_{\tau} \right) \le F(a\Delta + b\delta_w + c\delta_z) + (\text{small constant}).$$

Hence, for suitable small $\tau > 0$, the inequality in Definition 3.1 is satisfied with constants $a = \gamma, b = \alpha, c = \beta$, which satisfy $a + 2b + 2c = \gamma + 2\alpha + 2\beta < 1$ by assumption.

Step 3: By Theorem 3.3, T has a unique fixed point $w^* \in E$, which is the unique solution of the system

$$w(t) = \Phi(t) + \int_0^T K(t, s, w(s), w(T - s))\, ds.$$

5. Conclusion

In this paper, we introduced a new class of fixed point mappings termed enriched Hardy–Rogers F - contractions, which generalizes several known contraction principles by incorporating both nonlinear control functions and enriched distance terms. We established a fixed point theorem under this framework and demonstrated the convergence of the associated Krasnoselskij iterative process.

To illustrate the applicability of our result, we applied it to a new class of nonlinear integral equations involving symmetric feedback arising in models of recurrent learning and time-reflected systems. This application highlights the flexibility of the proposed contraction in handling nonlinearities and structural delays not captured by classical fixed point theories.

Our results not only unify and extend several existing fixed point results but also open up potential for further research into more complex integral and operator equations involving memory, feedback, or hybrid systems.

Transparency:

The author confirms that the manuscript is an honest, accurate, and transparent account of the study; that no vital features of the study have been omitted; and that any discrepancies from the study as planned have been explained. This study followed all ethical practices during writing.

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