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Hilbert and inner product spaces: Theory, visualization, and applications in machine learning

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Abstract: This study investigates the mathematical structure of Hilbert spaces, defined as complete inner product spaces, and their significance in both theoretical and applied contexts. We begin by exploring their foundational properties, including inner products, orthogonality, and completeness, which extend Euclidean geometric concepts to infinite-dimensional settings. Key mathematical tools, including the Cauchy–Schwarz inequality, triangle inequality, polarization identity, and Apollonius identity, are analyzed to highlight the analytical framework of Hilbert spaces and their relationship to normed spaces and Banach spaces. Then we examine practical applications in quantum mechanics, signal processing, and machine learning, where the inner product structure enables techniques like kernel methods, Support Vector Machines, and Principal Component Analysis. We provide MATLAB-based visualizations are provided, illustrating concepts such as projections and orthonormal expansions in computational contexts. This work integrates rigorous mathematical analysis with practical demonstrations, offering valuable insights for students and researchers in mathematics and data science.

Keywords: *Hilbert spaces, Inner product spaces, MATLAB visualization, Normed spaces, Principal component analysis (PCA), Reproducing kernel Hilbert space (RKHS), Support vector machines (SVM).*

1. Introduction

This paper investigates the structure and significance of Hilbert spaces, which are complete inner product spaces. Addressing the fundamental geometric concepts such as distance, angle, and orthogonality, familiar in finite dimensional Euclidean spaces, be extended to infinite dimensional settings. Hilbert spaces, defined as complete inner product spaces, provide a robust framework for this generalization, blending algebraic rigor with geometric intuition [1, 2]. Their completeness, ensuring that every Cauchy sequence converges within the space, makes them a cornerstone of both theoretical mathematics and applied sciences. This study aims to elucidate the properties of Hilbert spaces, demonstrate their analytical power through key mathematical tools, and showcase their applications in fields such as quantum mechanics, signal processing, and machine learning. By integrating rigorous theory with computational tools like MATLAB based visualizations of mathematics and data science.

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1.1. Historical Background and Literature Review

The concept of Hilbert spaces emerged from the need to generalize Euclidean geometry to abstract vector spaces, including those with infinite dimensions. At the core of this generalization is the inner product, a bilinear form that formalizes notions of length, angle, and orthogonality in a way that extends naturally from finite to infinite dimensions. A Hilbert space is an inner product space that is complete with respect to the norm induced by the inner product, ensuring convergence of Cauchy sequences a property critical for theoretical and applied contexts. This completeness distinguishes Hilbert spaces from general normed spaces, preserving geometric properties that enable powerful analytical techniques. Foundational work introducing sampling techniques in discrete reproducing kernel Hilbert spaces (RKHS) [3, 4].

The development of Hilbert spaces contributions by mathematicians like Schmidt [5] refined the geometric interpretation and terminology, laying the groundwork for modern functional analysis. Fundamental tools, such as the Cauchy–Schwarz inequality and Bunyakovsky inequality [6] triangle inequality [7] and polarization identity [8] form the analytical backbone of Hilbert space theory. These results not only provide deep insights into the structure of these spaces but also facilitate practical computations in diverse scientific domains.

Hilbert spaces have become indispensable in fields such as functional analysis, quantum mechanics, and partial differential equations [9]. The role of Hilbert spaces in quantum mechanics, focusing on their mathematical structure and applications [10]. In quantum mechanics, for instance, the state space of a quantum system is modeled as a Hilbert space, where the inner product governs probabilistic interpretations of wave functions. Similarly, in signal processing, Hilbert spaces underpin techniques for analyzing and reconstructing signals. More recently, the relevance of Hilbert spaces has surged in machine learning, where the inner product structure supports methods like Support Vector Machines (SVMs), Principal Component Analysis (PCA), and kernel methods [11-13]. These techniques leverage the geometry of high or infinite dimensional Hilbert spaces to address complex data analysis tasks, demonstrating the practical impact of abstract mathematical theory [8, 14].

This paper explores the theoretical foundations of Hilbert spaces and their inner product structure, supported by MATLAB based visualizations to illustrate abstract concepts. By examining applications in both traditional and modern contexts, we highlight the versatility of Hilbert spaces in bridging pure mathematics and computational practice. Our work aims to provide a unified perspective, making the theory accessible while emphasizing its relevance to contemporary challenges in science and technology [15, 16].

1.2. Preliminaries: Normed and Inner Product Spaces

This subsection introduces the foundational concepts and definitions necessary for understanding Hilbert spaces. A normed space is a vector space equipped with a norm, which induces a metric to measure distance. An inner product space is a normed space with an inner product, a bilinear form that defines geometric properties like orthogonality and angle. A Hilbert space is an inner product space that is complete with respect to the norm induced by the inner product, meaning every Cauchy sequence converges to a point in the space.

Normed Spaces: A normed on X is a real function $\|\bullet\|: X \rightarrow R$defined on X such that for any $x, y \in X$ and for all $\lambda \in K$. i. $\|x\| \geq 0$, ii. $\|x\| = 0 if and only if x = 0$, iii. $\|\lambda x\| = |\lambda| \|x\|$

iv. $\|x + y\| \leq \|x\| + \|y\| (Triangle inequality)$

A norm on X defines a metric d on X which is given by $d(x, y) = \|x - y\|; x, y \in X$and is called the metric induced by the norm. The normed space is denoted by $(X, \|\bullet\|)$ or simply by X

Inner Product Spaces: Let X be a vector space over the field K of real or complex. Then a mapping $\langle .,. \rangle: X \times X \rightarrow K$ is called an inner product of any $x, y \in X$ and for all $\alpha \in K$, which satisfies the following conditions:

i. $\langle x,x\rangle \geq 0 \; and \; \langle x,x\rangle = 0 \Leftrightarrow \mathrm{x} = 0$ ii. $\overline{\langle x,y\rangle} = \langle y,x\rangle$ iii. $\langle \alpha x,y\rangle = \alpha\langle x,y\rangle$
iv.$\langle x + y,z\rangle = \langle x,z\rangle + \langle y,z\rangle \quad \forall \; x,y,z \in X$

The vector space X together with inner product $\langle .,.\rangle$ is called an inner product or pre- Hilbert space. The inner product space is denoted by $(X, \langle .,.\rangle)$

The space $l(p)$: Let p_k be $+ve \; of \; R$ that is bounded above, s.t. $0 < p_k \leq \sup p_k = H < \infty$. Then the space $l(p)$ is defined as $l(p) = \{x = (x_k) : \sum_k |x_k|^{p_k} < \infty\}$. Also $d(x,y) = (\sum_k |x_k - y_k|^{p_k})^{\frac{1}{M}}$

The space l_p: Let $p_k = p$ be $+ve \; of \; R$ that is bounded above, s.t. $1 \leq \mathrm{p} < \infty$. Then the space l_p is defined as $l_p = \{x = (x_k) : \sum_k |x_k|^p < \infty\}$. Also $d(x,y) = (\sum_k |x_k - y_k|^p)^{\frac{1}{p}}$

The space $l_\infty(p)$: Let p_k be $+ve \; of \; R$ that is bounded above, s.t. $0 < p_k \leq \sup p_k = H < \infty$. Then the space $l(p)$ is defined as $l_\infty(p) = \left\{x = (x_k) : \sup_k |x_k|^{p_k} < \infty\right\}$. Also $d(x,y) = \sup_k |x_k - y_k|^{\frac{p_k}{M}}$

The space $l_\infty \; or \; l^\infty$: Let $p_k = p$ be $+ve \; of \; R$ that is bounded above, s.t. $1 \leq \mathrm{p} < \infty$. Then the space l_∞ is defined as $l_\infty = \left\{x = (x_k) : \sup_k |x_k|^p < \infty\right\}$. Also $d(x,y) = \sup_k |x_k - y_k|$

The space $c(p)$: Let p_k be $+ve \; of \; R$ that is bounded above, s.t. $0 < p_k \leq \sup p_k = H < \infty$. Then the space $c(p)$ is defined as $l(p) = \{x = (x_k) : |x_k - l|^{p_k} < \infty \quad \forall \; l \in C\}$. Also $d(x,y) = \sup_k |x_k - y_k|^{\frac{p_k}{M}}$

The space c: Let $p_k = p$ be $+ve \; of \; R$ that is bounded above, s.t. $1 \leq \mathrm{p} < \infty$. Then the space c is defined as

$c = \{x = (x_k) : |x_k - l|^p < \infty\}$. Also, $d(x,y) = \sup_k |x_k - y_k|$

The space $c_0(p)$: Let p_k be $+ve \; of \; R$ that is bounded above, s.t. $0 < p_k \leq \sup p_k = H < \infty$. Then the space $c(p)$ is defined as $l(p) = \{x = (x_k) : |x_k|^{p_k} < \infty \quad \}$. Also, $d(x,y) = \sup_k |x_k - y_k|^{\frac{p_k}{M}}$

The space c_0 : Let $p_k = p$ be $+ve \; of \; R$ that is bounded above, s.t. $1 \leq \mathrm{p} < \infty$. Then the space c_0 is defined as $c_0 = \{x = (x_k) : |x_k|^p < \infty \quad \}$. Also, $d(x,y) = \sup_k |x_k - y_k|$,

1.3. Arrangement of the Article

The structure of this paper is organized as follows. Section 2 presents the parallelogram law and examines its importance in identifying Hilbert spaces. Section 3 focuses on the development and formulation of Hilbert spaces. Section 4 highlights general applications of Hilbert spaces. Section 5 the discussion to their use in machine learning. Section 6 explores applications of inner product spaces, followed by Section 7, which considers their role in machine learning. Section 8 integrates the applications of both inner product and Hilbert spaces in machine learning contexts. Section 9 provides a conceptual perspective on human reasoning in machine learning through the lens of these spaces. Finally, **Section 10** offers concluding remarks.

2. Parallelogram Law and Characterization of Hilbert Space

In classical geometry, the parallelogram law asserts that the total of the squares of a parallelogram's diagonals is equal to the sum of the squares of its four sides. This concept has a significant counterpart in functional analysis. In particular, within normed vector spaces, an analogous identity holds in inner product spaces. This identity is universally valid in inner product spaces and plays a vital role in identifying when a normed space originates from an inner product. It leads to a significant result that a normed linear space is a Hilbert space if and only if its norm satisfies the parallelogram law. If a normed space satisfies this condition, it is possible to define an inner product that induces the norm, effectively transforming the space into an inner product space. If, in addition, the space is complete under this norm, it qualifies as a Hilbert space. Thus, the parallelogram law not only captures a fundamental geometric principle but also serves as a powerful analytical tool for recognizing Hilbert spaces within the wider context of normed vector spaces.

2.1. Theorem (Parallelogram Law)

For any two elements x and y belonging to an inner product space X, then $\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2$

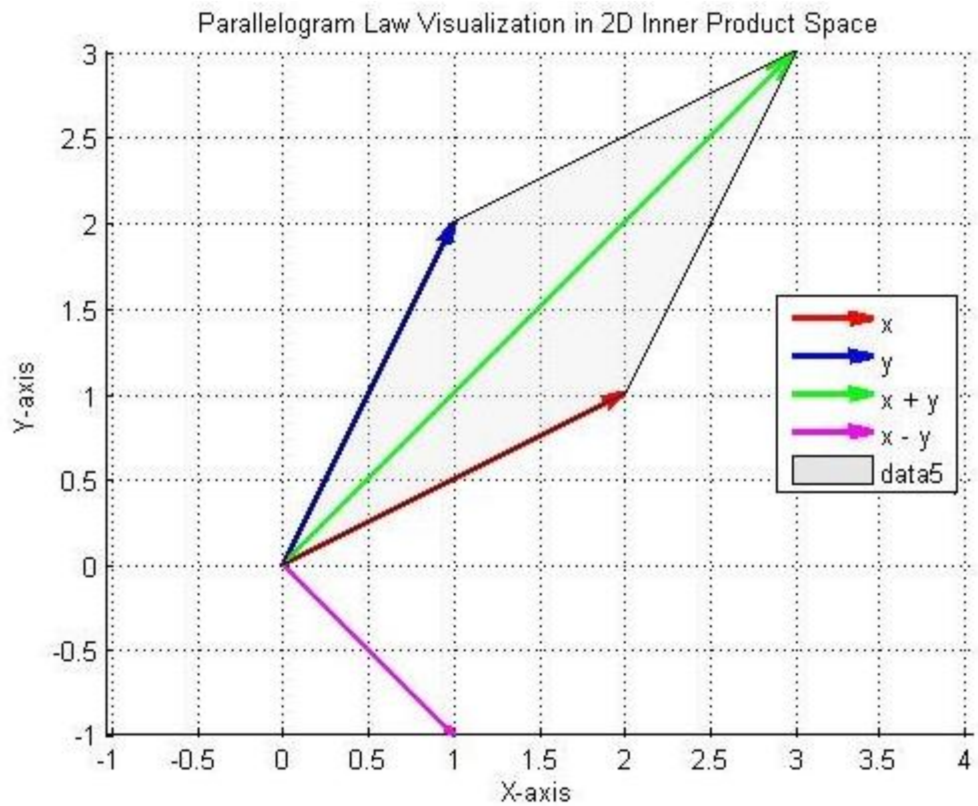

Figure 1.
MATLAB Visualization of the Parallelogram Law.

The Figure 1, provides a geometric illustration of the Parallelogram Law in a 2D inner product space. Two vectors x (red) and y (blue) originate from the origin and form the adjacent sides of a parallelogram. Their vector sum $x + y$ (green) and difference $x - y$ (magenta) serve as the diagonals. The shaded parallelogram visually demonstrates that the sum of the squared lengths $\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2$,confirming the Parallelogram Law in inner product spaces.

2.2. Theorem

Prove that the space l_p is not a Hilbert space for $p \neq 2$

2.2.1. Geometrical Explanation

Why the space l_p is not a Hilbert space for $p \neq 2$. To geometrically understand and why the space l_p is not a Hilbert space unless $p = 2$, we must look at the shape of unit balls and the notion of angles in these spaces.

i. Hilbert Spaces and Inner Product Geometry

In a Hilbert space i. e. l_2, geometry behaves similarly to Euclidean space: a) A well-defined inner product spaces, b) Define angles and orthogonality, c) The unit ball (set of points with norm ≤ 1) is round — a perfect circle (in 2D) or sphere (in higher dimensions), d) The parallelogram law holds: it characterizes how vector lengths and angles interact.

ii. l_2 Spaces and Geometry of Unit Balls

Let us consider the unit ball in R^2 under different l_p norms:

P	**Unit Ball Shape**	**Geometry Behavior**
P=1	Diamond-shaped	Corners, no smoothness
P=2	Perfect circle	Euclidean, inner product exists
$P = \infty$	Square	Flat sides, angles not well-defined
$p \neq 2$	Smooth but not circular	Geometry is distorted

iii. Why This Matters for Hilbert Spaces

A Hilbert space is more than just a complete normed space such that project one vector onto another, and angles make sense.

For $p \neq 2$, these geometric tools break down: a) No proper projection theorem, b) No orthogonal decomposition, c) No true "angles" between vectors.

The geometry of l_p space for $p \neq 2$ is not Euclidean: The unit balls are not round, the norm does not IPS, and l_p fails to be a Hilbert space unless p=2.

Proof: Let us take $x = (1,1,0,0,...) \in l_p and\ x = (1,-1,0,0,...) \in l_p$.

Then $x + y = (2,0,0,...) \quad and\ x - y = (0,2,0,0,...)$. We have

$$\|x\| = \left(\sum_{k=1}^{n} |x_k|^p\right)^{\frac{1}{p}} = (|1|^p + |1|^p + 0 + 0 + ... + 0)^{\frac{1}{p}} = 2\ and\ \|y\| = (|1|^p + |-1|^p + 0 + 0 + ... + 0)^{\frac{1}{p}} = 2$$

Also $\|x + y\| = (|2|^p + 0 + 0 + ... + 0)^{\frac{1}{p}} = 2\ and\ \|x - y\| = (0 + |2|^p + 0 + 0 + ... + 0)^{\frac{1}{p}} = 2$

So that $\|x + y\|^2 + \|x - y\|^2 = 2^2 + 2^2 = 8\ and\ 2\|x\|^2 + 2\|y\|^2 = 2\left(2^{\frac{2}{p}} + 2^{\frac{2}{p}}\right)$

If p=2, then $2\|x\|^2 + 2\|y\|^2 = 2(2 + 2) = 8$

Thus, for p=2, the parallelogram law $\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2$ is satisfied.

This implies that the space l_2 is a Hilbert space.

When $p \neq 2$, the parallelogram law does not hold. Consequently, the space l_p is not a Hilbert space for values of p other than 2.

To visualize the failure of the parallelogram law in MATLAB for l_p spaces when $p \neq 2$, we can create a 2D plot showing how the quantity $D(p) = \|x + y\|_p^2 + \|x - y\|_p^2$, varies with p and compare it to the value 4 (which is required by the parallelogram law). $For\ p = 2, D(p) = 4;\ for\ p \neq 2, D(p) \neq 4.$

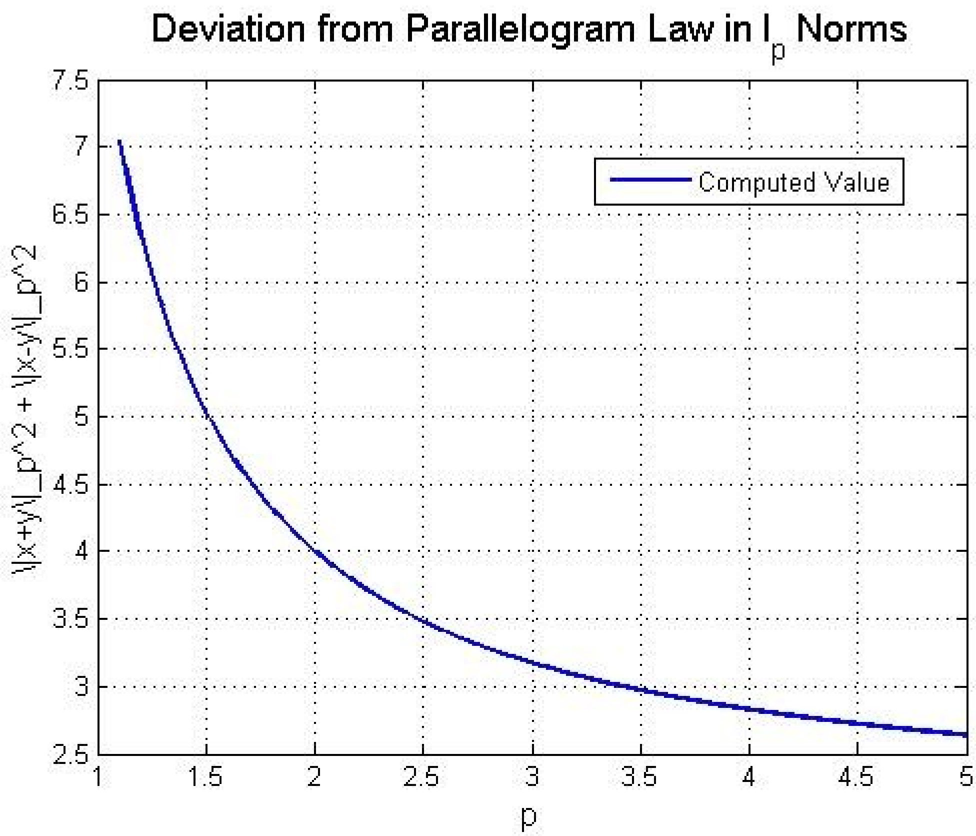

Figure 2.
MATLAB Visualization of the Deviation from Parallelogram Law in l_p Norms.

This Figure 2, shows that l_p spaces deviate most from the Parallelogram Law when p is far from 2, and the deviation gradually decreases as p approaches 2, highlighting that only when $p = 2$ does the norm come from an inner product.

2.3. Theorem (Cauchy Schwartz inequality)
If X is an inner product space and $x, y \in$ X, then $|< x, \mathrm{y} >| \le \|x\|\|y\|$.

2.3.1. Geometrical Interpretation of the Cauchy–Schwarz Inequality

The Cauchy–Schwarz inequality provides a geometric bound on the inner product of two vectors. In Euclidean space, the inner product $< x, \mathrm{y} >$ can be written as: $< x, \mathrm{y} > = \|x\|\|y\|. cos\theta$.
Using this, the Cauchy–Schwarz inequality becomes: $|< x, \mathrm{y} >| \le \|x\|. \|y\|. cos\theta \le \|x\|. \|y\|$.

since $|\ cos\ \theta\ | \le 1$for all real angles θ. Equality occurs when $\theta = 0\ or\ \pi$, meaning $x\ and\ y$ point in the same or opposite directions and are thus linearly dependent. In geometric terms, this inequality implies that the projection of one vector onto another cannot be longer than the product of their lengths. It ensures that the cosine of the angle between two vectors always lies between -1 and 1, preserving the familiar geometric structure even in abstract inner product spaces.

2.3.2. MATLAB Visualization of the Cauchy–Schwarz Inequality

The Cauchy–Schwarz inequality is a key result stating that the absolute value of the inner product of two vectors does not exceed the product of their norms. It is crucial for proving many other results in functional analysis, Hilbert spaces, and vector geometry. Here, Plots two vectors $x \, and \, y$, the projection of $x \, onto \, y$ and Computes both sides of the Cauchy–Schwarz inequality.

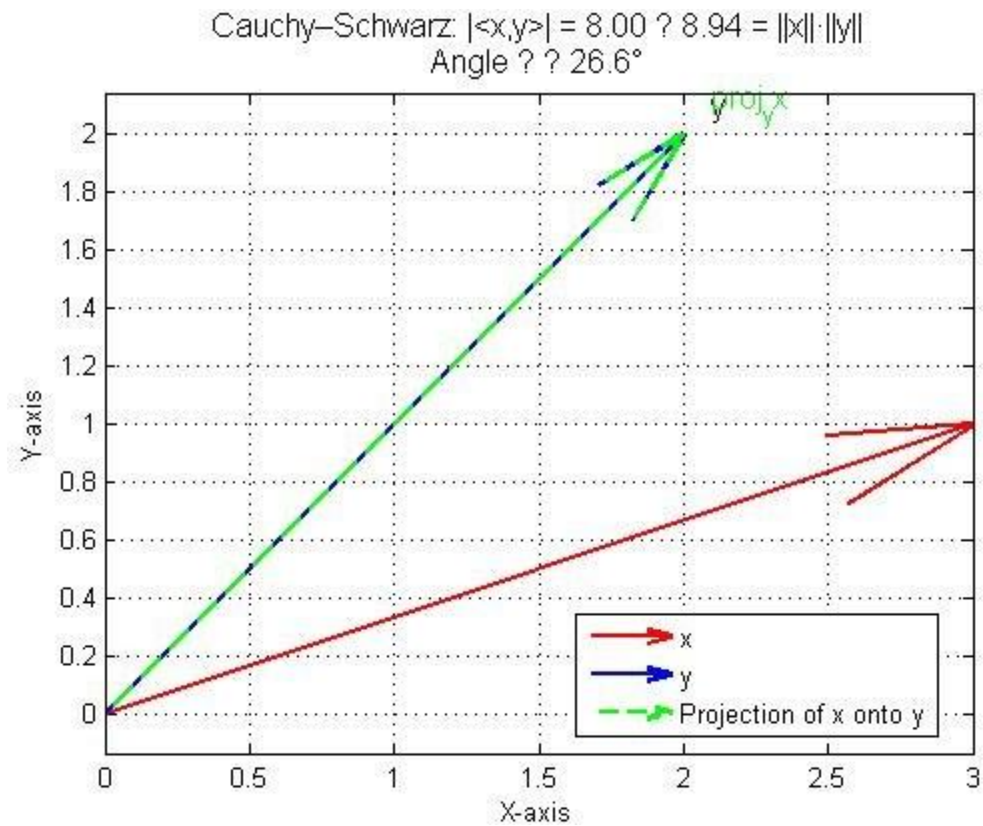

Figure 3.
MATLAB Visualization of the Cauchy–Schwarz inequality.

The Figure 3, illustrates the Cauchy-Schwarz inequality and the projection of vector x onto vector y in a 2D space. The red vector x extends from (0,0) to approximately (2.5, 1), and the blue vector y extends to (2, 2). The green vector shows the projection of x onto y. The inequality $|< x, y >| \approx 8.00 \, to \, 8.94$ is close to $\|x\|. \|y\|$.

2.4. Theorem

The space C [a, b] is not an inner product space, hence not a Hilbert space.

2.4.1. Geometrical Explanation

Why C [a, b] is not a Hilbert Space?

Here, C [a, b] is not a Hilbert space because its norm is not induced by an inner product, and it is not complete under the L_2 norm, which is essential for Hilbert spaces.

i. Geometry in Hilbert Spaces (i. e. L^2[a, b])

In Hilbert spaces: a) The norm comes from an inner product, b) Define angles and orthogonality, c) There is a clear Pythagorean geometry: $\|x + y\|^2 = \|x\|^2 + \|y\|^2 \;\; if \;\; x \perp y$, d) A project of a function onto a subspace (like Fourier series projection), e) The geometry behaves like Euclidean space, but infinite-dimensional.

ii. Geometry in C [a, b]
The space C [a, b], with the supremum norm: $\|f\|_\infty = \sup_{t\in[a,b]} |f(t)|$
iii. Geometrical Difference via Unit Balls

In 2D geometry, you can visualize normed spaces by looking at **unit balls** (the set of vectors/functions with norm 1).

Space	**Norm**	**Unit Ball Shape**	**Geometry Type**
${l_2}^2$	$\|x\|_2$	Circle	Inner product space
${l_\infty}^2$	$\|x\|_\infty$	Square	Not inner product
C [a, b]	$\|f\|_\infty$	Infinite-dimensional cube-like	No angles, no projection

In C [a, b], the geometry is dominated by uniform height across the interval, not by averaging as in L^2. This makes the "roundness" (essential for inner products) absent.

The space C [a, b] lacks the Euclidean-like geometry of Hilbert spaces: The unit ball is not round, there is no inner product, so no angles, no orthogonality, and no projections and even under the L^2 inner product, it's not complete, so you can't use Hilbert space geometry reliably.

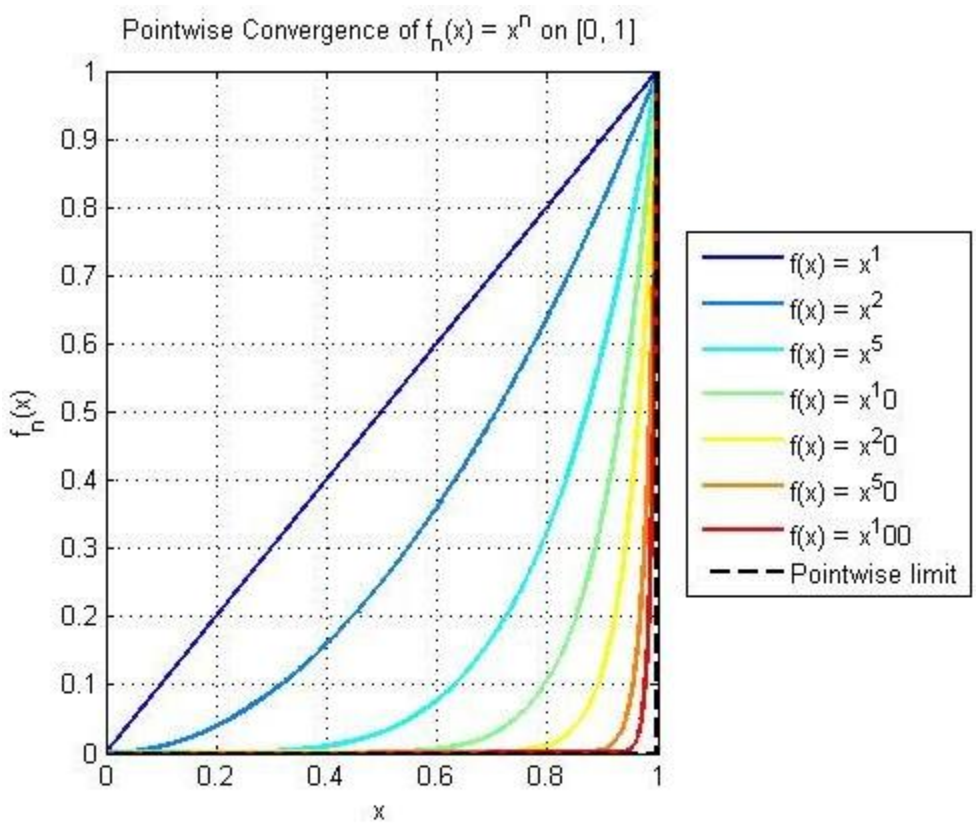

Figure 4.
MATLAB Visualization for the space C [a, b].

The Figure 4, illustrates the pointwise convergence of the sequence $f_n(x) = x_n$ *are in* $C[0,1]$. It shows curves for $x^1(blue), x^2(cyan), x^5\ (green), x^{10}(yellow), x^{50}(orange), and\ x^{100}\ (red),$ with the pointwise limit (black dashed line) approaching 0 for $x \in [0,1)$ *and* 1 *at* $x = 1$.

3. Result and Discussion

Hilbert spaces, as complete inner product spaces, extend Euclidean geometry to infinite dimensions while preserving key properties like orthogonality and completeness. They underpin important theorems and enable machine learning methods such as SVMs and kernel PCA through Reproducing Kernel Hilbert Spaces. MATLAB visualizations help illustrate these geometric concepts in data analysis and learning.

3. 1. Theorem

Prove that the Euclidean space $\mathbb{R}^n$ is a Hilbert space.

Proof: The space R^n is a Hilbert space with inner product defined by

$\langle a,b\rangle = a_1b_1 + a_2b_2 + \ . \ . \ . \ + a_nb_n$

(1)

Where $a = (a_i) = (a_1, \ a_2, \ . \ . \ . \ , a_n) \in R^n$ and $b = (b_i) = (b_1, \ b_2, \ . \ . \ . \ , b_n\ ) \in R^n$

In fact, from (1) we obtain, $\|a\| = \langle a,a\rangle^{\frac{1}{2}} = \langle {a_1}^2 + {a_2}^2 + \ldots + {a_n}^2\rangle^{\frac{1}{2}}$

and $d(a,b) = \|a-b\| = \langle a-b, a-b\rangle^{\frac{1}{2}} = \langle (a_1-b_1)^2 + (a_2-b_2)^2 + \ldots + (a_n-b_n)^2\rangle^{\frac{1}{2}}$

Firstly, we will prove that the Euclidean space $\mathbb{R}^n$ is complete.

We know that the metric of $\mathbb{R}^n$ is $d(a,b) = \ (\sum_{i=1}^{n}(a_i-b_i)^2)^{\frac{1}{2}}$

Where $a_i = a_1\ , a_2, \ldots \ldots . . , a_n$ and $b_i = b_1, b_2 \ldots \ldots \ldots , b_n\ , \ \forall\ a_i, \ b_i \in \mathbb{R}$

Let $< a_n >$ be a Cauchy sequence in $\mathbb{R}^n$. For every $\epsilon > 0 \ \exists\ m, r \in \mathbb{N}$

s.t. $d(a_m, a_r) = \left(\sum_{i=1}^{n}\left(a_i^{(m)} - a_i^{(r)}\right)^2\right)^{\frac{1}{2}} < \epsilon \qquad \forall\ m, r \in \mathbb{N}$ (2)

Both sides squaring in(i) , then we get $\left(a_i^{(m)} - a_i^{(r)}\right)^2 < \epsilon^2 \qquad \forall\ i = 1,2,3, \ldots \ldots \ldots , n \Rightarrow \left|a_i^{(m)} - a_i^{(r)}\right| < \epsilon$

Fixed i, $(1 \le i \le n)$, and $< a_i^1, a_i^2 \ldots \ldots \ldots \ldots >$ is a Cauchy sequence of $\mathbb{R}$. So it converges i.e. $a_i^m \to a$ as $m \to \infty$. Using this n limits, we define $a = (a_1\ , a_2, \ldots \ldots . . , a_n)$. Clearly $a \in \mathbb{R}^n$, From(2), $a_r \to a$ $as\ r \to \infty$, then we have $d(a_m\ , \ a) \le \epsilon\ , m > N$

So, $\lim_{m\to\infty} a_m = a$. Hence $\mathbb{R}^n$ is complete.

If n=3, then (1) gives $\langle a,b\rangle = a.b = a_1b_1 + a_2b_2 + a_3b_3$

of $a = (a_1, \ a_2, \ a_3) \in R$ and $b = (b_1, \ b_2, \ b_3) \in R$ and the orthogonality $\langle a,b\rangle = a.b = 0$

This concept is consistent with the fundamental idea of perpendicularity, meaning that two vectors are orthogonal if their inner product is zero.

Therefore, the Euclidean space $\mathbb{R}^n$ is a Hilbert space.

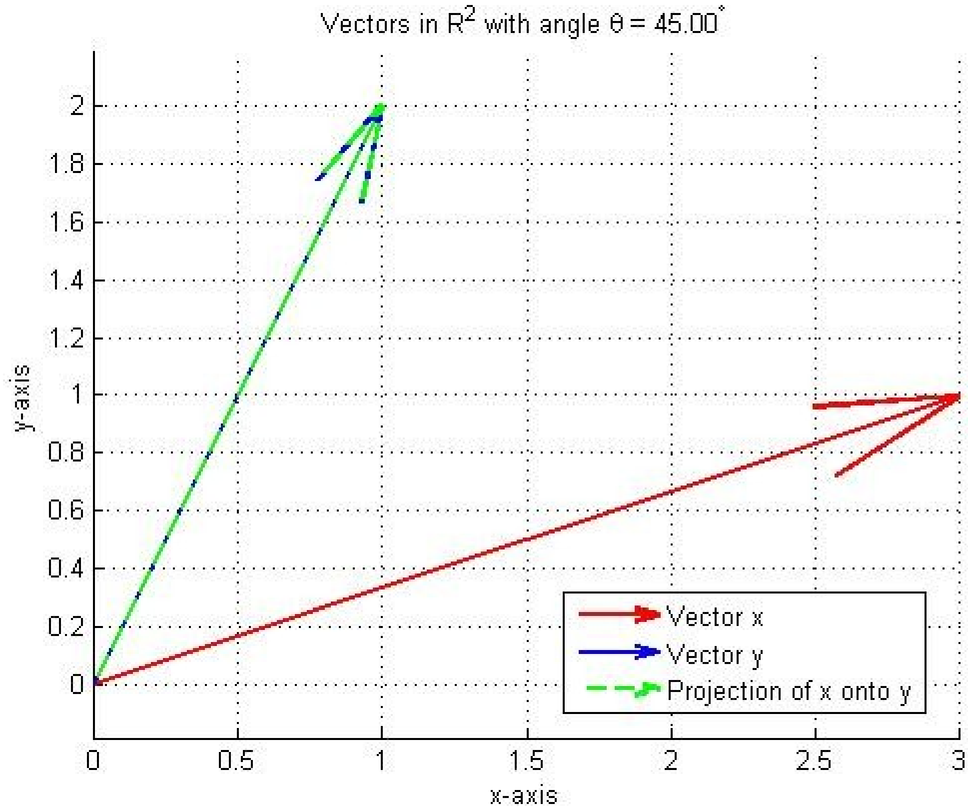

Figure 5.
MATLAB Visualization for the Euclidean space in R^2.

The Figure 5, shows vectors in R^2 with an angle of 45°. The red vector x extends from $(0,0)$ to approximately $(3,1)$, the blue vector y extends to about $(2,2)$, and the green vector represents the projection of x onto y, ending at around $(1.5,1.5)$.

Example (Orthogonality in R^n): In R^2, let $x = (1,0)$, $y = (0,1)$. Then $\langle x, y\rangle = 1 \cdot 0 + 0 \cdot 1 = 0$, so x and y are orthogonal, representing perpendicular vectors.

Example (Completeness in R^n): Consider a Cauchy sequence (x_k) $in\ R^2$, where $x_k = (1 - \frac{1}{k}, \frac{1}{k})$. $For\ k > m$, the distance is $\|x_k - x_m\| = \sqrt{(\frac{1}{k} - \frac{1}{m})^2 + (\frac{1}{m} - \frac{1}{k})^2}$, which approaches 0 as $m, k \to \infty$. The sequence converges to $(1,0) \in R^2$, confirming completeness.

3.2. Theorem

Prove that the unitary space C^n is a Hilbert space.

Proof: The space C^n is a Hilbert space with inner product defined by

$\langle a, b\rangle = a_1\overline{b_1} + a_2\overline{b_2} + \ .\ .\ .\ + a_n\overline{b_n}$

(3)

Where $a = (a_i) = (a_1,\ a_2,\ .\ .\ .\ , a_n) \in C^n$ and $b = (b_i) = (b_1,\ b_2,\ .\ .\ .\ , b_n\ ) \in C^n$

In fact, from (3) we obtain, $\|a\| = \langle a, a\rangle^{\frac{1}{2}} = \left(a_1\overline{b_1} + a_2\overline{b_2} + \ldots + a_n\overline{b_n}\right)^{\frac{1}{2}} = \left(|a|_1{}^2 + |a_2|^2 + \ldots + |a_n|^2\right)^{\frac{1}{2}}$ and from this the unitary metric defined by

$$d(a, b) = \|a - b\| = \langle a - b, a - b\rangle^{\frac{1}{2}} = \langle |a_1 - b_1|^2 + |a_2 - b_2|^2 + \ldots + |a_n - b_n|^2\rangle^{\frac{1}{2}}$$

Firstly, we will prove that the unitary space C^n is complete.

We know that the metric in C^n is $d(a,b) = \left(\sum_{i=1}^{n} |a_i - b_i|^2\right)^{\frac{1}{2}}$
Where $a_i = a_1, a_2, \dots\dots, a_n$ and $b_i = b_1, b_2 \dots\dots, b_n$
Let $< x_n >$ be a Cauchy sequence in C^n. For every $\epsilon > 0\ \exists\ m, n \in \mathbb{N}$

s. t. $d(a_m, a_n) = \left(\sum_{i=1}^{n}\left(a_i^{(m)} - a_i^{(n)}\right)^2\right)^{\frac{1}{2}} < \epsilon \quad \forall\ m, n \in \mathbb{N}$
(4)

Both sides squaring in(4) , then we get $\left(a_i^{(m)} - a_i^{(n)}\right)^2 < \epsilon^2 \quad \forall\ i = 1,2,3, \dots\dots, n \Rightarrow \left|a_i^{(m)} - a_i^{(n)}\right| < \epsilon$

Fixed i, $(1 \le i \le n)$, the sequence $< x_i^1, x_i^2 \dots\dots >$ is a Cauchy sequence of C. So it converges i.e. $a_i^m \to a$ as $m \to \infty$. Using this n limits, we define $a = (a_1, a_2, \dots\dots, a_n)$. Clearly $a \in C^n$, From(2), $a_n \to a\ as\ n \to \infty$, then we have $d(a_m, a) \le \epsilon, m > N$

So, $\lim_{m\to\infty} a_m = a$. Hence the unitary space C^n is complete

If n=3, then (3) gives $\langle a, b\rangle = a.\overline{b} = a_1\overline{b_1} + a_2\overline{b}_2 + a_3\overline{b_3}$
of $a = (a_1, a_2, a_3) \in C$ and $b = (b_1, b_2, b_3) \in C$ and the orthogonality $\langle a, b\rangle = a.b = 0$.

This concept is consistent with the fundamental idea of orthogonality, meaning that two vectors are orthogonal if their inner product is zero. Therefore, the unitary space C^n is a Hilbert space.

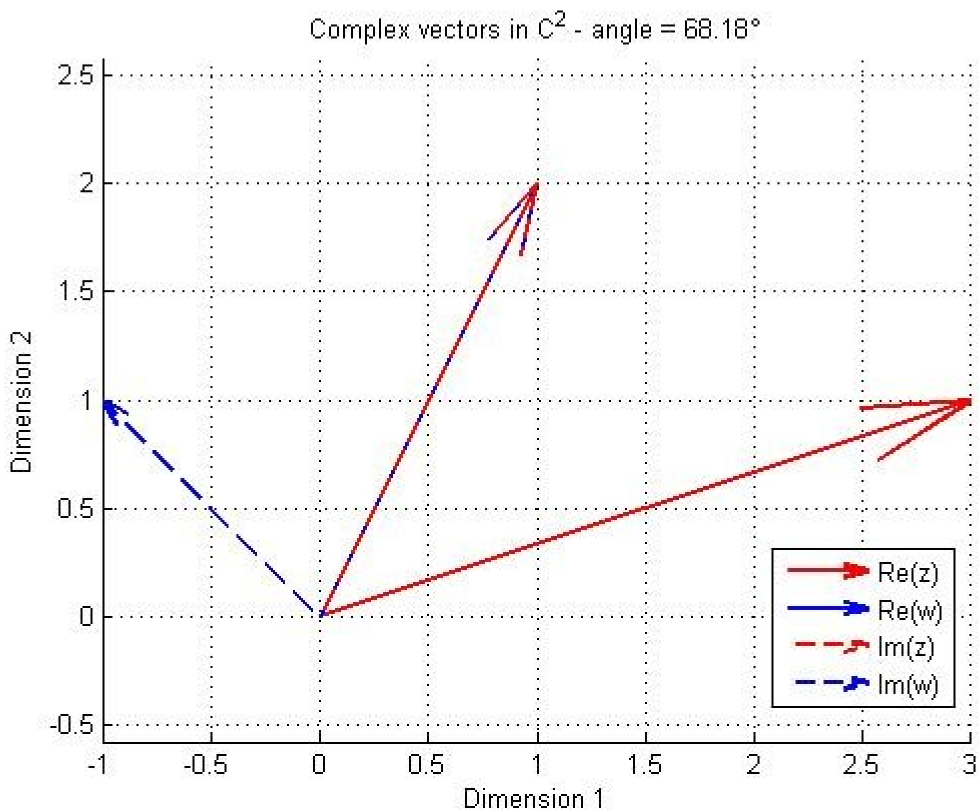

Figure 6.
MATLAB Visualization for the unitary space in C^2.

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The Figure 6, depicts complex vectors in C^2 with an angle of 68.18^0. The red solid line represents the real part of z (Re(z)), the blue dashed line represents the real part of w (Re(w)), the red dashed line represents the imaginary part of z (Im(z)), and the blue solid line represents the imaginary part of w (Im(w)), showing their components in a 2D plane.
Example (Orthogonality in C^n): In C^2, let $x = (1, i), y = (i, -1)$. Then $\langle x, y\rangle = 1 \cdot (-i) + i \cdot 1 = 0$. So x *and* $(i, 1)$ are orthogonal.

3.3. Theorem
Prove that the space l_2 is a Hilbert space.
Proof: The space l_2 is a Hilbert space with inner product defined by
$\langle a, b\rangle = a_1\overline{b_1} + a_2\overline{b_2} + \ . \ . \ . + a_n\overline{b_n}$
(5)
Where $a = (a_i) = (a_1, \ a_2, \ . \ . \ . \ , a_n) \in l_2$ and $b = (b_i) = (b_1, \ b_2, \ . \ . \ . \ , b_n \ ) \in l_2$

In fact, from (3) we obtain, $\|a\| = \langle a, a\rangle^{\frac{1}{2}} = \left(a_1\overline{b_1} + a_2\overline{b_2} + \ldots + a_n\overline{b_n}\right)^{\frac{1}{2}} = \left(|a|_1{}^2 + |a_2|^2 + \ldots + |a_n|^2\right)^{\frac{1}{2}}$
and from this the unitary metric defined by

$$d(a, b) = \|a - b\| = \langle a - b, a - b\rangle^{\frac{1}{2}} = \langle |a_1 - b_1|^2 + |a_2 - b_2|^2 + \ldots + |a_n - b_n|^2\rangle^{\frac{1}{2}}$$

Firstly, we will prove that the unitary space l_2 is complete.

We know that the metric in C^n is $d(a, b) = \ (\sum_{i=1}^{n} |a_i - b_i|^2)^{\frac{1}{2}}$
Where $a_i = a_1 \, , a_2, \ldots \ldots \ldots , a_n$ and $b_i = b_1, b_2 \ldots \ldots \ldots , b_n$
Let $< x_n >$ be a Cauchy sequence in l_2 . For every $\epsilon > 0 \ \exists \ m, n \in \mathbb{N}$

s. t. $$d(a_m, a_n) = \left(\sum_{i=1}^{n}\left(a_i^{(m)} - a_i^{(n)}\right)^2\right)^{\frac{1}{2}} < \epsilon \qquad \forall \ m, n \in \mathbb{N}$$
(6)

Both sides squaring in(4) , then we get $\left(a_i^{(m)} - a_i^{(n)}\right)^2 < \epsilon^2 \quad \forall \ i = 1,2,3, \ldots \ldots \ldots , n \ \Rightarrow \left|a_i^{(m)} - a_i^{(n)}\right| < \epsilon$
Fixed i, $(1 \le i \le n)$, the sequence $< x_i^1, x_i^2 \ldots \ldots \ldots \ldots >$ is a Cauchy sequence of R. So it converges i.e. $a_i^m \to a$ as $m \to \infty$. Using this n limits, we define $a = (a_1 \, , a_2, \ldots \ldots \ldots , a_n)$. Clearly $a \in l_2$, From(2), $a_n \to a \ as \ n \to \infty$, then we have $d(a_m \, , \ a) \le \epsilon \, , m > N$
So, $\lim_{m\to\infty} a_m = a$. Hence the unitary space l_2 is complete
If n=3, then (3) gives $\langle a, b\rangle = a.\overline{b} = a_1\overline{b_1} + a_2\overline{b}_2 + a_3\overline{b_3}$
of $a = (a_1, \ a_2, \ a_3) \in R$ and $b = (b_1, \ b_2, \ b_3) \in R$ and the orthogonality $\langle a, b\rangle = a.b = 0$.
This concept is consistent with the fundamental idea of orthogonality, meaning that two vectors are orthogonal if their inner product is zero. Therefore, the space l_2 is a Hilbert space.

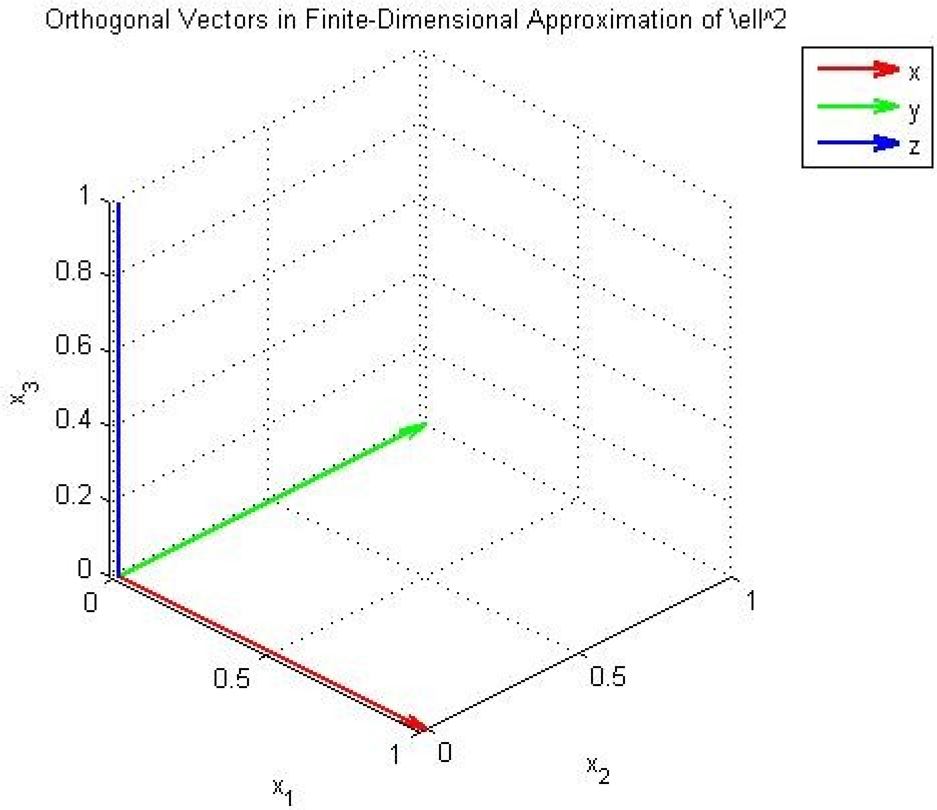

Figure 7.
MATLAB Visualization of l^2 Truncations

The Figure 7, shows that l^2geometrically by projecting infinite-dimensional concepts onto 2D or 3D subspaces of R^n. The inner product, norm, and orthogonality behave analogously to how they do in Euclidean space. This visualization preserves the core Hilbert space geometry: norm, angle, projection, and orthogonality. Also, this MATLAB code visualizes three orthogonal sequences (finite truncations) as vectors in R^3.

Example (Orthogonality in l_2): Let $x = (1, 0, 0, \ldots), y = (0, 1, 0, \ldots)$. Then $\langle x, y \rangle = 1 \cdot 0 + 0 \cdot 1 + 0 = 0$, So x and y are orthogonal.

Example (Completeness in l_2): Consider $x_k = \left(\frac{1}{k}, \frac{1}{k^2}, \ . \ . \ . \frac{1}{k^n}, 0, \ . \ . \ . \right)$. Compute $\|x_k - x_m\|^2 = \sum_{n=1}^{min(k,m)} \left|\frac{1}{k^n} - \frac{1}{m^n}\right|^2 + \sum_{n=\min(k,m)+1}^{max(k,m)} \left|\frac{1}{k^n}\right|^2$, which approaches 0. The limit $(0, 0, \ . \ . \ .) \in l_2$, confirming completeness

3.4. Theorem (Polarization Identity and Hilbert Space)

Let B be a complex Banach space, and suppose the norm $\|\cdot\|$ on B satisfies the parallelogram law. Define a map $\langle \cdot,\cdot \rangle$ on $B \times B$ by $4\langle p, q \rangle := \| p + q \|^2 - \| p - q \|^2 + i \| p + iq \|^2 - i \| p - iq \|^2$.
Then $\langle \cdot,\cdot \rangle$ is an inner product, and hence B, being complete, is a Hilbert space.
Proof:
To show that B is a Hilbert space, then the map $\langle \cdot,\cdot \rangle$ satisfies the axioms of an inner product on a complex vector space.

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3.4.1. Positivity and Definiteness

Set $q = p$. Then $4\langle p,p\rangle = \| p + q \|^2 - \| p - q \|^2 + i \| p + iq \|^2 - i \| p - iq \|^2$
$= \| 2p \|^2 - 0 + i \| p(1 + i) \|^2 - i \| p(1 - i) \|^2$
$= 4 \| p \|^2 + i \mid 1 + i \mid^2 \| p \|^2 - i \mid 1 - i \mid^2 \| p \|^2$
$= 4 \| p \|^2 + 2i \| p \|^2 - 2i \| p \|^2 = 4 \| p \|^2$.
Thus, $\langle p,p\rangle = \| p \|^2 \geq 0$ and $\langle p,p\rangle = 0$ implies $\| p \| = 0$, hence $p = 0$.

3.4.2. Conjugate Symmetry

Take the complex conjugate of the polarization identity:
$4\overline{\langle p,q\rangle} = \| p + q \|^2 - \| p - q \|^2 - i \| \underline{p + iq} \|^2 + i \| p \underline{- iq} \|^2$
By symmetry and norm properties: $4\overline{\langle p,q\rangle} = 4\langle p,p\rangle \Rightarrow \overline{\langle p,q\rangle} = \langle p,p\rangle$

3.4.3. Linearity in the First Argument

Let $p, q, r \in B$. We have, $\langle p + q, r\rangle = \langle p,r\rangle + \langle q,r\rangle$.
Then we applying the polarization identity to $p + q$ and r, expanding each term, and using the parallelogram law repeatedly s.t. $\| p + q + r \|^2 - \| p + q - r \|^2 = (\| p + r \|^2 - \| p - r \|^2) + (\| q + r \|^2 - \| q - r \|^2)$
similarly for the imaginary parts, $\langle p + q, r\rangle = \langle p,r\rangle + \langle q,r\rangle$

3.4.4. Homogeneity in the First Argument

We verify for scalar $\alpha \in C$,

- $Case\ \alpha = n \in N: Use\ induction\ on\ n.$
- $Case\ \alpha = -1: Substitute\ \ -p\ in\ the\ identity\ and\ show\ \langle -p,q\rangle = -\langle p,q\rangle$
- $Case\ \alpha \in Q: Write\ \alpha = \frac{s}{t}, and\ verify\ using\ linearity\ and\ scalar\ multiplication.$
- $Case\ \alpha = i: Replace\ p\ with\ ip\ in\ the\ identity\ and\ simplify: \langle ip,q\rangle = i\langle p,q\rangle$
- $Case\ \alpha = a + ib \in C: Use\ linearity: \langle \alpha p,q\rangle = \langle ap + ibp,q\rangle = a\langle p,q\rangle + ib\langle p,q\rangle = \alpha\langle p,q\rangle$

$\Rightarrow \langle \alpha p,q\rangle = \alpha\langle p,q.$

The map $\langle\cdot,\cdot\rangle$ satisfies are all the properties of an inner product. Since B is already a Banach space (i.e., complete), thus, B satisfied all the conditions of an inner product space. Therefore B is an inner product space and hence B is a Hilbert space.

3.4.5. Interpretation

- The polarization identity extracts full geometric information (lengths and angles) from norms.
- By measuring lengths of carefully combined vectors, it reconstructs the inner product the fundamental tool for angle, projection, and orthogonality in Hilbert spaces.
- The MATLAB plot shows how real and imaginary combinations of vectors contribute to the full inner product.

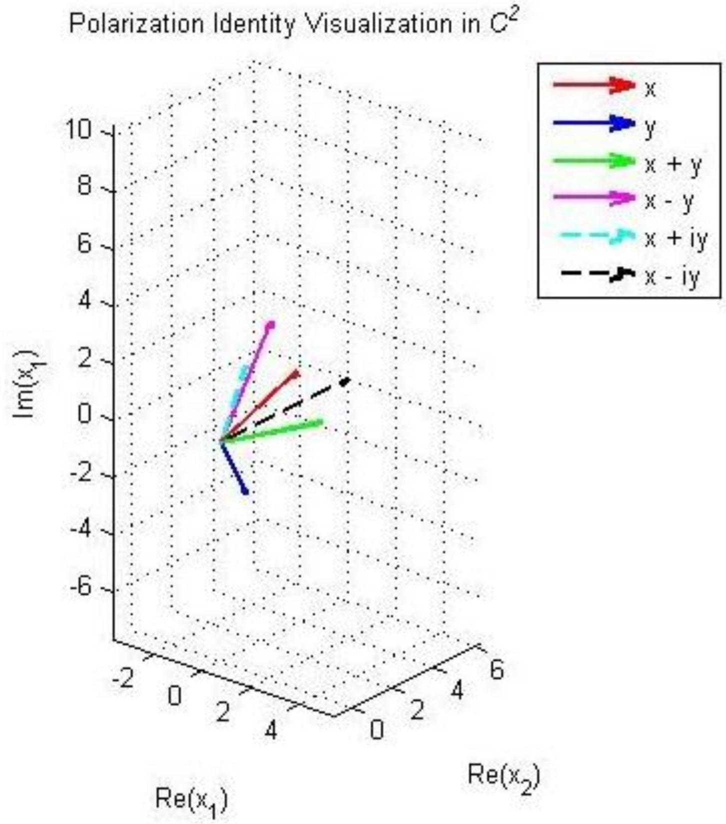

Figure 8.
MATLAB Visualization of the Polarization Identity.

The Figure 8, visualizes the polarization identity in C^2showing vectors x (red), y (blue), $\mathrm{x} + \mathrm{y}$ (green), $\mathrm{x} + \mathrm{iy}$ (magenta), and $\mathrm{x} - \mathrm{iy}$(cyan) originating from the origin in a 3D space with axes $Re(x_1), Re(x_2), and\ Im\ (x)$, illustrating the relationship between complex vector components.

3.5. Discussion

The preceding results confirm that $\mathbb{R}^n, C^n\ and\ l_2$are Hilbert spaces by virtue of possessing both an inner product structure and completeness. These two features are essential in extending geometric intuition from Euclidean spaces to more abstract settings. Orthogonality allows for decomposition and projection of elements a mechanism at the heart of many computational techniques such as principal component analysis (PCA) in machine learning. Completeness guarantees that limits of Cauchy sequences remain within the space, providing a foundation for the convergence of iterative numerical algorithms. Together, these properties support a broad range of applications, from the mathematical formulation of quantum systems to signal representation and processing in engineering.

4. Applications of Hilbert Spaces

4.1. Quantum Mechanics

In quantum mechanics, the state of a physical system is described by a vector in a complex Hilbert space. Observable physical quantities correspond to self-adjoint (Hermitian) operators acting on this space. The inner product on the Hilbert space is crucial for interpreting measurement results probabilistically: the squared magnitude of the inner product between two state vectors represents the

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probability of transitioning from one state to another. This framework offers a precise mathematical foundation for the inherently probabilistic behavior observed in quantum systems.

4.2. Signal and Image Processing

Hilbert spaces are fundamental in Fourier analysis and the theory of wavelets. Signals are often modeled as elements of L^2 spaces, and decomposing signals into orthonormal bases (e.g., sine/cosine in Fourier transforms) is a core technique.

4.3. Machine Learning and Data Science

Reproducing Kernel Hilbert Spaces (RKHS) are used in support vector machines and Gaussian processes. These spaces enable kernel methods that efficiently operate in high-dimensional or infinite-dimensional feature spaces.

4.4. Numerical Analysis and PDEs

Hilbert spaces provide the setting for the variational formulation of partial differential equations. The famous Lax-Milgram theorem, for instance, guarantees the existence and uniqueness of solutions under certain conditions in a Hilbert space setting.

5. Applications of Hilbert Spaces in Machine Learning

5.1. Kernel Methods and Reproducing Kernel Hilbert Spaces (RKHS) [17, 18].

Kernel methods implicitly map data into high-dimensional Hilbert spaces without computing the coordinates directly. This allows for: Non-linear classification using linear techniques in transformed spaces. Efficient implementation via the kernel trick: $K(x, y) = \langle \phi(x), \phi(y) \rangle$

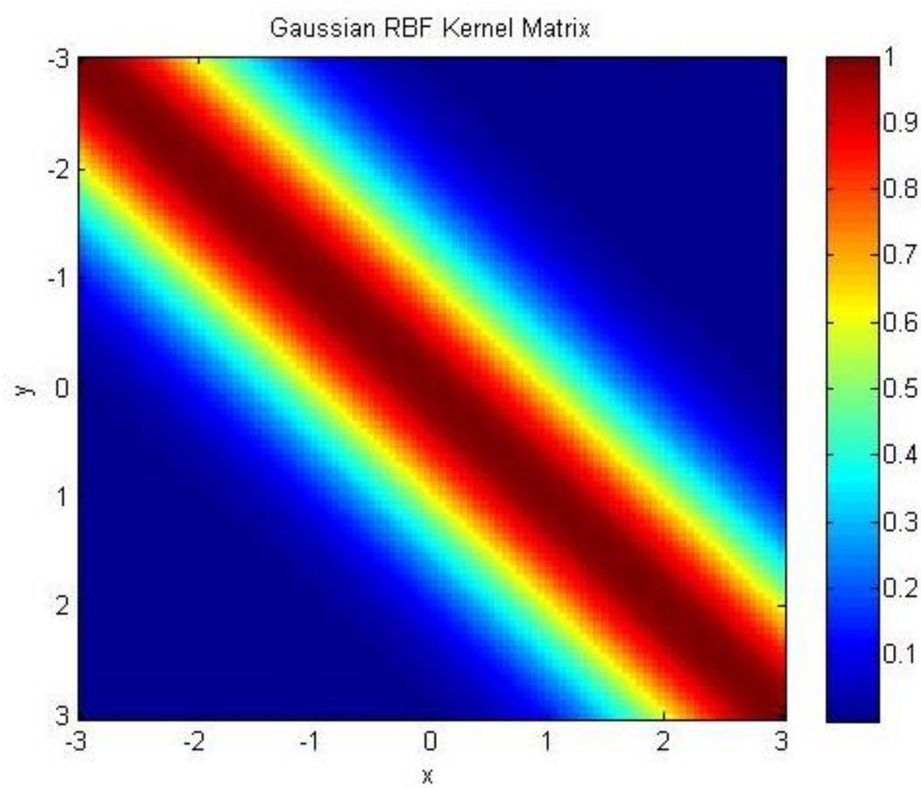

Figure 9.
MATLAB Implementation of the Visualizing a Gaussian (RBF) Kernel and Feature Mapping.

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The Figure 9, shows a Gaussian RBF (Radial Basis Function) Kernel Matrix, visualized as a heatmap. The x and y axes range from -3 to 3, representing pairs of data points. The color gradient, ranging from blue $(0.1)\ to\ red\ (1.0)$, indicates the similarity between these points, with warmer colors showing higher similarity. The diagonal line of red indicates maximum similarity where points are compared to themselves, while the similarity decreases as points move away from the diagonal.

5.2. Support Vector Machines (SVMs)

SVMs find the optimal hyperplane in a Hilbert space to separate data. The use of kernels enables non-linear boundaries in the original space while maintaining computational feasibility. SVMs is supervised algorithms applied to classification and regression problems. They identify the hyperplane that maximizes the margin between different classes in a transformed feature space. For datasets that are not linearly separable, kernel functions are used to project data into higher-dimensional spaces, enabling the discovery of a linear boundary.

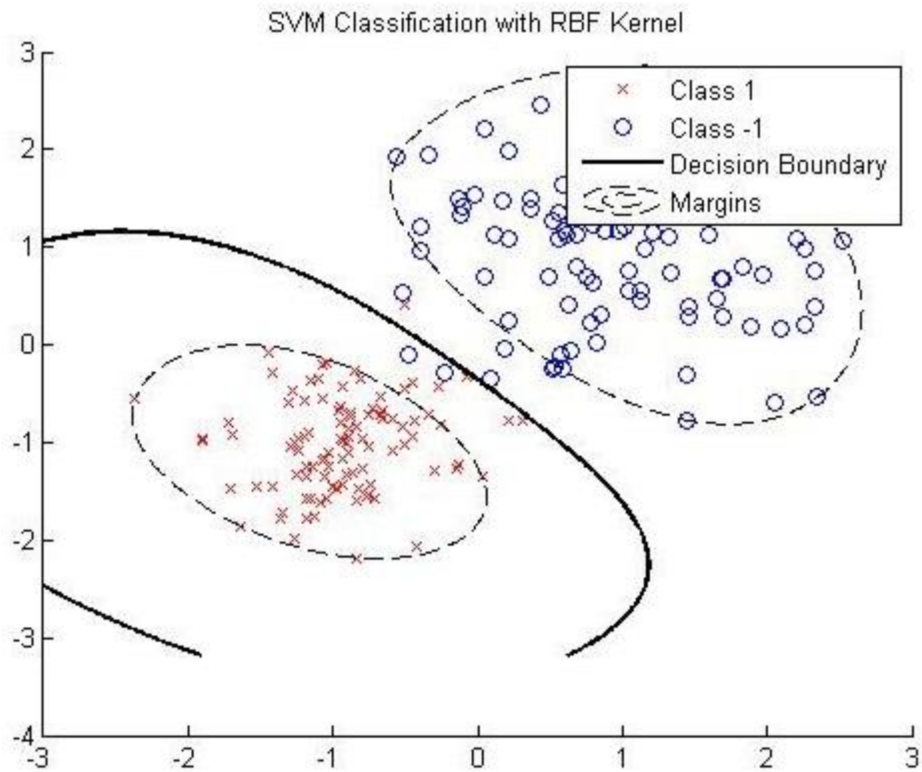

Figure 10.
MATLAB Implementation: Visualizing a SVM Classification.

The Figure 10, illustrates SVM (Support Vector Machine) classification using an RBF (Radial Basis Function) kernel. It shows two classes: Class $1\ (\text{red}\ 'x')\ \text{and Class}\ -1\ (\text{blue}\ 'o')$, separated by a nonlinear decision boundary (black curve). The dashed lines represent the margins, which define the boundary's width, with support vectors (data points closest to the boundary) influencing its position.

5.3. Principal Component Analysis (PCA)

PCA is a statistical technique designed to reduce the dimensionality of a dataset by transforming it into a smaller set of variables while retaining most of the original information. This is achieved by identifying new axes, known as principal components, which are linear combinations of the original features. These components are ranked based on how much of the data's variation they capture. The first captures the greatest amount of variation, followed by the second, and so on. By projecting the data onto these principal components, PCA helps simplify complex data, making it easier to explore, visualize, and interpret.

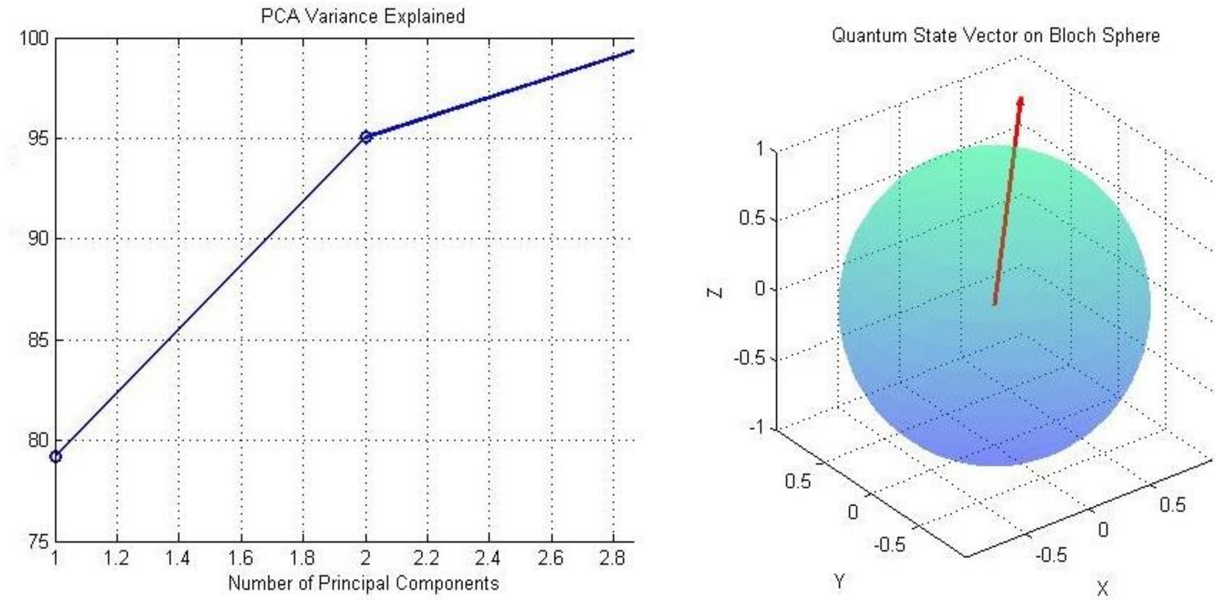

Figure 11.
MATLAB Implementation: Visualizing a PCA Classification with quantum state vectors as unit vectors on a 3D Bloch sphere.

The Figure11, consists of two parts. The left graph, "PCA Variance Explained," shows a plot where the y-axis represents the percentage of variance explained (75-100%) and the x-axis represents the number of principal components (1 to 2.8). The curve rises steeply, indicating that most variance is captured with a few components. The right image, "Quantum State Vector on Bloch Sphere," depicts a 3D sphere with a red vector pointing from the origin to a point near the top, illustrating a quantum state vector's position in a Bloch sphere representation.

5.4. Orthonormal Basis in L^2 (Fourier Sine and Cosine Functions) and Interactive Function Visualizations (e.g., Fourier Basis in L^2 Space)

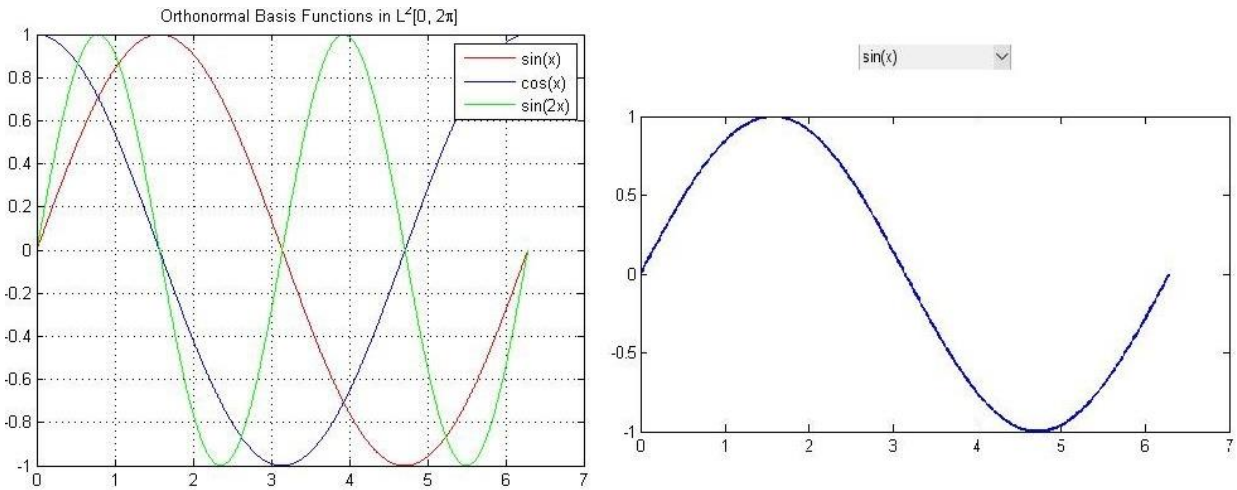

Figure 12.
MATLAB Implementation: This visualizes how sine and cosine functions can act as orthonormal **bases** in the $L^2[0,2\pi]$ Hilbert space.

The Figure 12, shows orthonormal basis functions in $L^2[0,\ 2\pi]$on the left, with three curves: sin(x) in red, cos(x) in green, and sin(2x) in blue, oscillating between -1 and 1 over the interval [0, 7]. The right plot zooms in on sin(x), displaying its standard sinusoidal wave with a single peak and trough within [0, 7], ranging from -1 to 1.

6. Applications of Inner Product Spaces

Inner product spaces play a pivotal role in both pure and applied mathematics due to their geometric structure and analytical properties. Their applications span across various disciplines, including:

6.1. Quantum Mechanics

In quantum theory, the state of a physical system is represented by a vector in a complex Hilbert space (a complete inner product space). Observables such as position and momentum are modeled as linear operators on these spaces. The inner product allows the computation of probabilities and expectations, forming the core of quantum measurement theory.

6.2. Signal Processing and Fourier Analysis

Inner product spaces provide the foundation for signal decomposition techniques such as the Fourier transform. Signals can be represented as sums of orthogonal basis functions, and inner products are used to compute the coefficients in these expansions. This is critical in filtering, compression, and noise reduction.

6.3. Computer Graphics and Geometry

In 3D graphics, the inner product (or dot product) is used to determine angles between vectors, shading, and projection operations. This underpins rendering techniques and physical simulations.

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6.4. Statistics and Principal Component Analysis (PCA)

In multivariate statistics, PCA uses the inner product to measure variance and correlation. Data is projected onto orthogonal components (eigenvectors of the covariance matrix), which are computed using inner product operations.

6.5. Functional Analysis and PDEs

Many boundary value problems for partial differential equations are solved within inner product spaces using methods such as the Galerkin or Ritz method. These rely on projecting infinite-dimensional problems into finite-dimensional subspaces using orthogonality conditions.

6.6. Signal Decomposition Using Orthonormal Basis (Fourier Analysis)

Signals can be represented as linear combinations of orthonormal functions. The coefficients are obtained via inner products.

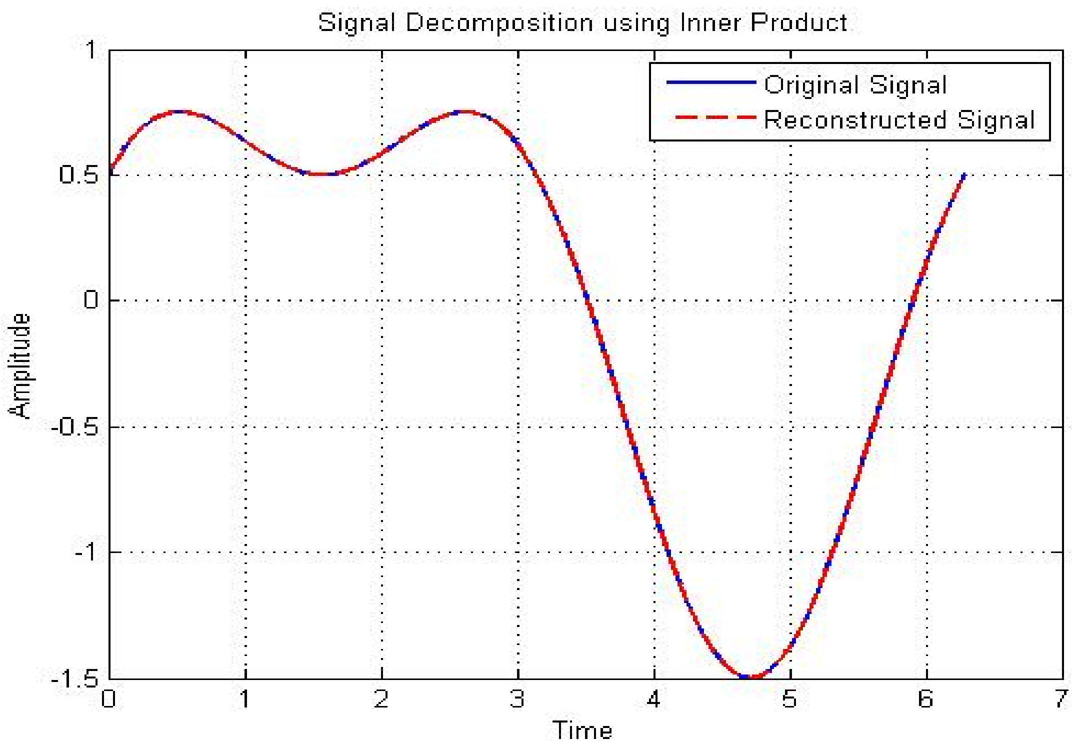

Figure 13.
MATLAB Implementation: Visualizing a Signal Decomposition Using Inner Product.

The Figure 13, illustrates signal decomposition using the inner product, comparing an original signal (blue) and its reconstructed signal (red). Both signals vary in amplitude from -1.5 to 1 over time (0 to 7), showing a similar pattern with peaks and troughs, indicating the reconstruction closely matches the original.

7. Applications of Inner Product Spaces in Machine Learning

Inner product spaces form a mathematical foundation for many algorithms in machine learning, offering a framework to measure similarity, define geometry in feature spaces, and facilitate learning in high-dimensional settings. Their properties enable both theoretical insights and practical implementations in various learning paradigms.

7. 1. Similarity Measures and Kernel Methods

The inner product is frequently used to quantify the similarity between vectors. In classification and clustering, similarity functions guide decisions about groupings or class membership. Kernel methods extend this idea by computing inner products in transformed feature spaces, often through kernel functions. This is central to algorithms like the Support Vector Machine (SVM), where the decision boundary is constructed using inner products between data points in a high-dimensional (possibly infinite-dimensional) Reproducing Kernel Hilbert Space (RKHS).

7.2. Principal Component Analysis (PCA)

PCA is a dimensionality reduction technique that relies on inner products to compute the covariance matrix and its eigenvectors. These eigenvectors form an orthogonal basis that captures the directions of maximum variance in the data. By projecting data onto these directions using inner products, PCA simplifies the feature space while preserving key information, improving both interpretability and computational efficiency.

7.3. Neural Networks and Optimization

Although neural networks are nonlinear models, inner products still appear in the computation of neuron activations, especially in fully connected layers. Each neuron computes a weighted inner product between the input vector and a weight vector, followed by a non-linear activation. Additionally, gradient-based optimization methods used to train networks rely on inner product-based notions such as gradient descent directions and orthogonal projections in parameter space.

7.4. Recommendation Systems

Matrix factorization techniques for collaborative filtering in recommendation systems use inner products to model user item interactions. In such systems, the predicted rating of a user for an item is computed as the inner product of the user's and item's latent feature vectors. Also, this manuscript builds on our earlier analytic work applying Sylow's theorems to small composite orders [19, 20].

8. Applications of Inner Product and Hilbert Spaces in Machine Learning

8.1. Inner Product Spaces in Machine Learning

An inner product space provides a way to measure angles and distances between vectors. This structure underlies several machine learning concepts, especially where similarity, projection, and orthogonality are essential. Also, the machine learning concepts are like Similarity Measures, PCA & Data Compression, Regression & Optimization

8.2. Hilbert Spaces in Machine Learning

A Hilbert space is a complete inner product space, allowing infinite-dimensional extensions of vector space methods. In machine learning, Hilbert spaces (especially RKHS) are used to generalize linear algorithms to nonlinear contexts through kernel functions, Functional Regression, Spectral Learning Methods.

Table 1.
Comparison in Inner Product and Hilbert Spaces in Machine Learning.

Feature	Inner Product Space	Hilbert Space (Complete Inner Product Space)
Dimensionality	Finite-dimensional	Can be infinite-dimensional
Completeness	Not necessarily complete	Always complete
Use in ML	Feature similarity, projections, PCA, regression	Kernel methods, Gaussian processes, RKHS, functional learning
Examples in ML	Linear regression, PCA, cosine similarity, k-means	SVM (with kernel), Gaussian processes, spectral learning
MATLAB Tool Use	dot, eig, matrix algebra	Kernel computation, functional inner products
Mathematical Framework	Euclidean geometry	Functional analysis

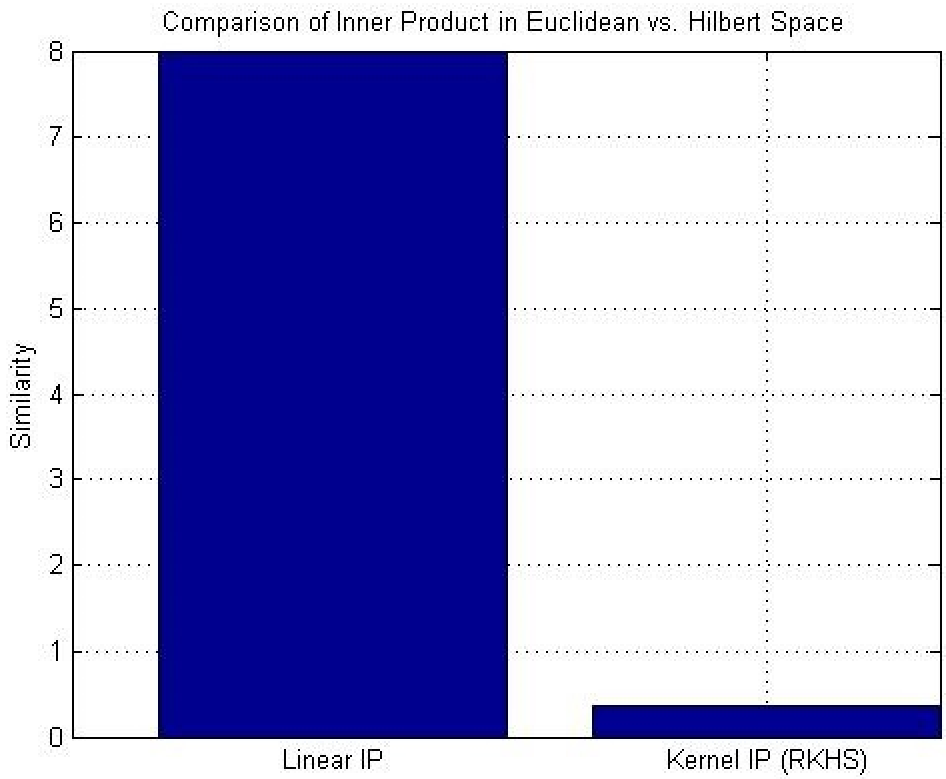

Figure 14.
MATLAB Simulation: Inner Product vs. RKHS (Hilbert Space via Kernel).

This simulation compares linear inner product similarity and kernel-based similarity (in RKHS).

The Figure 14, compares the similarity of inner products in Euclidean versus Hilbert space. It shows a bar chart with two categories: "Linear IP" and "Kernel IP (RKHS)". The "Linear IP" bar reaches approximately 7 on the similarity scale, while the "Kernel IP (RKHS)" bar is much lower, around 1, indicating a significant difference in similarity measures between the two methods.

Table 2.
The Comparison of Inner Product Spaces vs. Hilbert Spaces in Machine Learning.

Feature	**Inner Product Space (e.g., Linear SVM)**	**Hilbert Space (e.g., Kernel SVM with RBF)**
Mathematical Foundation	Vector space with finite-dimensional inner product	Complete inner product space (possibly infinite-dim)
Key Use in ML	Linear classification, regression, PCA	Nonlinear classification, kernel methods
Decision Boundary Shape	Linear (hyperplane)	Nonlinear (curved, flexible)
Flexibility	Limited to linear separability	Can handle complex, nonlinear patterns
Similarity Measure	Euclidean dot product	Kernel-induced inner product (e.g., Gaussian kernel)
SVM Kernel Function in MATLAB	'linear'	'rbf', 'polynomial', or custom kernels
Performance on Nonlinear Data	Poor	Excellent
Computational Cost	Low	Higher (due to kernel matrix computation)

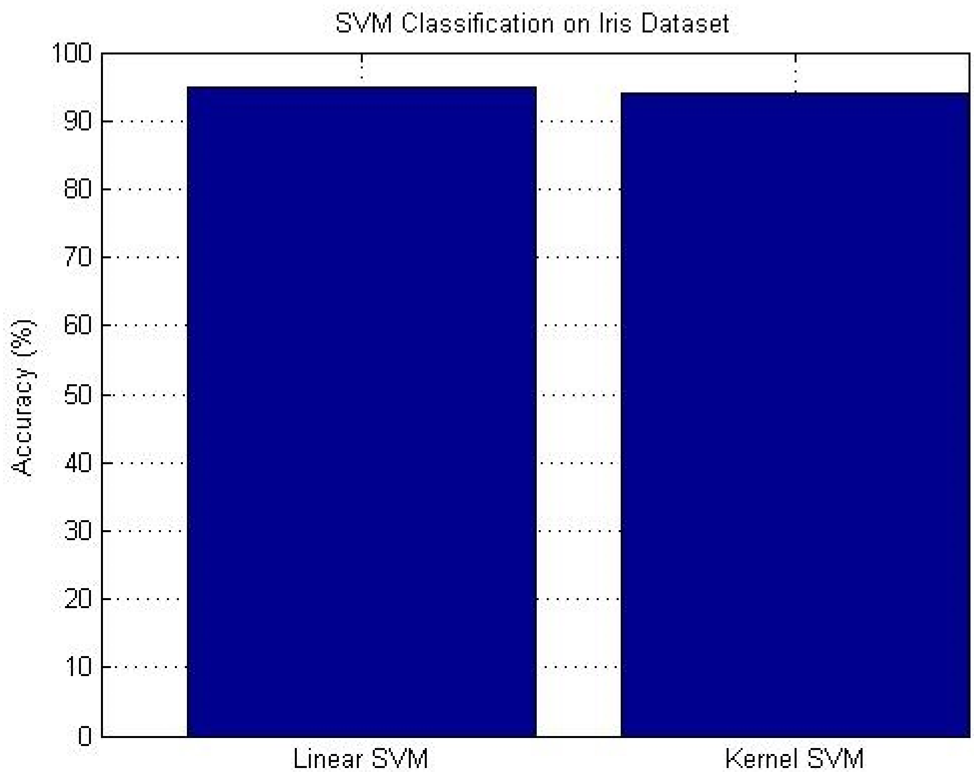

Figure 15.
MATLAB Simulation: Linear vs. Kernel SVM (Using Synthetic Data).

We generate nonlinear data, then apply both linear and kernel SVMs to compare their performance—demonstrating the practical contrast between an inner product space and a Hilbert space.

The Figure 15, shows a bar chart comparing the accuracy of SVM classification on the Iris dataset using Linear SVM and Kernel SVM. Both methods achieve an accuracy of approximately 90-100%, with no significant difference between them.

9. Conceptual Overview: Human Thinking in ML with Inner Product and Hilbert Spaces

We visualize the Inner Product Space as the human thinking linearly—making decisions based on straight-line similarity or dot product intuition. And the Hilbert Space (via RKHS) as the human thinking abstractly or nonlinearly, visualizing patterns beyond what's visible—like imagination mapping reality to a higher-dimensional space[21, 22].

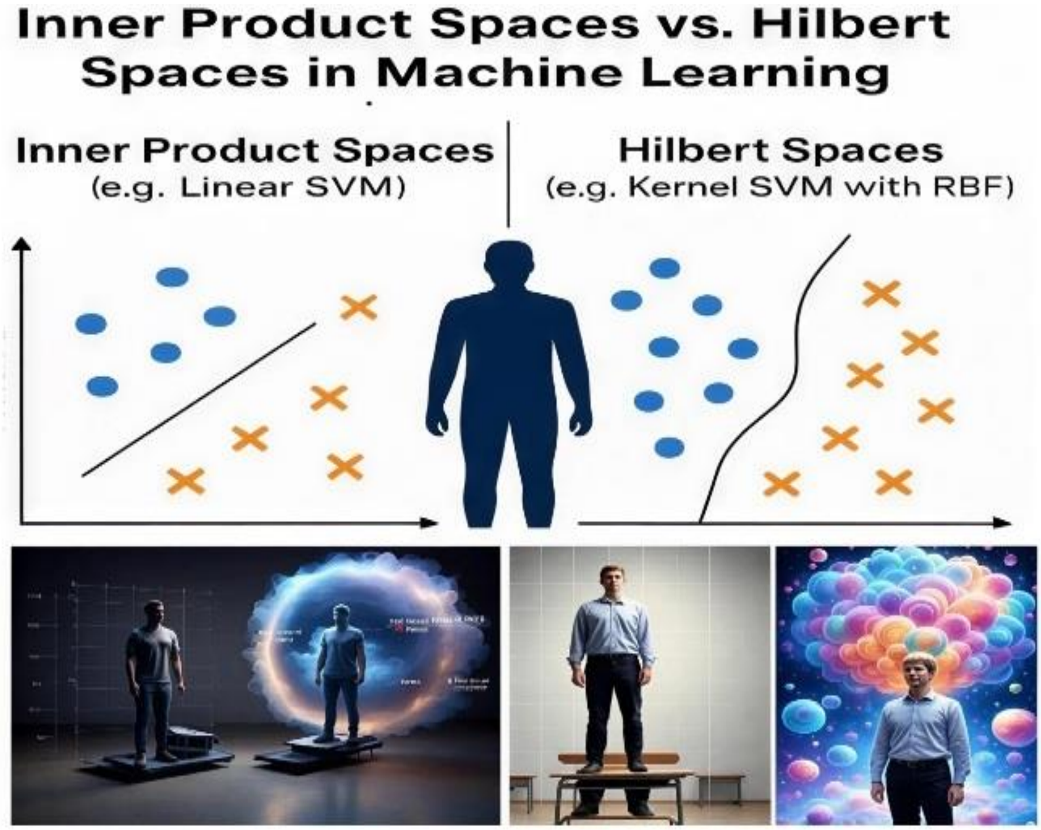

Figure 16.
Human Thinking in ML with Inner Product and Hilbert Spaces.

The Figure 16, emphasizes how human reasoning evolves:

- In the inner product space, the approach is direct, simple, and linear.
- In the Hilbert space, reasoning becomes abstract, enabling the recognition of nonlinear patterns—similar to how kernel methods work in machine learning.

10. Limitations and Future Work

Although this study provides both conceptual insight and computational demonstrations of Hilbert and inner product spaces with MATLAB based illustrations relevant to machine learning several limitations persist, offering opportunities for further research and refinement.

10.1. Theoretical Limitations

The present analysis is limited to separable Hilbert spaces and bounded linear operators. Advanced topics such as non-separable Hilbert spaces, unbounded operators, and spectral theory were not explored. These areas, however, are crucial for a deeper understanding in fields like quantum mechanics, mathematical physics, and infinite-dimensional learning theory, and warrant further investigation in future work.

10.2. Methodological and Modeling Gaps

This study focuses on foundational concepts and techniques and does not incorporate more sophisticated machine learning models such as deep neural networks, support vector machines with kernel methods, or recent developments in deep kernel learning. In particular, the practical utilization of reproducing kernel Hilbert space (RKHS) frameworks within contemporary deep learning models has not been addressed. Bridging this gap could be a valuable direction for future research.

11. Conclusion

This study demonstrates that Hilbert and inner product spaces generalize the geometric and algebraic structure of Euclidean spaces to infinite dimensional settings, providing a rigorous framework for projections, orthogonality, and completeness. By formally analyzing $\mathbb{R}^n, C^n$ and l_2, illustrating the parallelogram law, and applying these concepts in PCA, SVMs, quantum mechanics, and signal processing, we highlighted the practical and theoretical significance of Hilbert spaces. While the focus was on separable Hilbert spaces and illustrative computational examples, the work lays a foundation for exploring advanced topics such as Hilbert Schmidt operators, RKHS-Banach duality, and applications in large-scale machine learning or quantum computing. Overall, Hilbert spaces emerge as a central mathematical framework that bridges theory with practical algorithms, enabling robust problem-solving across mathematics, data science, and physics.

Transparency:

The authors confirm that the manuscript is an honest, accurate, and transparent account of the study; that no vital features of the study have been omitted; and that any discrepancies from the study as planned have been explained. This study followed all ethical practices during writing.

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